

Polynomial representations of C^* -algebras and their applications

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Abstract

This is a sequel to our paper on nonlinear completely positive maps and dilation theory for real involutive algebras, where we have reduced all representation classification problems to the passage from a C^* -algebra \mathcal{A} to its symmetric powers $S^n(\mathcal{A})$, resp., to holomorphic representations of the multiplicative $*$ -semigroup (\mathcal{A}, \cdot) . Here we study the correspondence between representations of \mathcal{A} and of $S^n(\mathcal{A})$ in detail. As $S^n(\mathcal{A})$ is the fixed point algebra for the natural action of the symmetric group S_n on $\mathcal{A}^{\otimes n}$, this is done by relating representations of $S^n(\mathcal{A})$ to those of the crossed product $\mathcal{A}^{\otimes n} \rtimes S_n$ in which it is a hereditary subalgebra. For C^* -algebras of type I, we obtain a rather complete description of the equivalence classes of the irreducible representations of $S^n(\mathcal{A})$ and we relate this to the Schur–Weyl theory for C^* -algebras. Finally we show that if $\mathcal{A} \subseteq B(\mathcal{H})$ is a factor of type II or III, then its corresponding multiplicative representation on $\mathcal{H}^{\otimes n}$ is a factor representation of the same type, unlike the classical case $\mathcal{A} = B(\mathcal{H})$.

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1 Introduction

A real *seminormed involutive algebra* (or *$*$ -algebra*) is a pair (\mathcal{A}, p) , consisting of a real associative algebra \mathcal{A} endowed with an involutive antiautomorphism $*$ and a submultiplicative seminorm p satisfying $p(a^*) = p(a)$ for $a \in \mathcal{A}$. Our project, started in [BN12], eventually aims at a systematic understanding of unitary representations of unitary groups $U(\mathcal{A})$ of real seminormed involutive algebras (\mathcal{A}, p) . If \mathcal{A} is unital, its *unitary group* is

$$U(\mathcal{A}) := \{a \in \mathcal{A} : a^*a = aa^* = \mathbf{1}\},$$

and if \mathcal{A} is non-unital, then $U(\mathcal{A}) := U(\mathcal{A}^1) \cap (\mathbf{1} + \mathcal{A})$, where $\mathcal{A}^1 = \mathcal{A} \oplus \mathbb{R}\mathbf{1}$ is the unitization of \mathcal{A} . Typical examples we have in mind are algebras of the form $\mathcal{A} = C^\infty(X, \mathcal{B})$ for a Banach $*$ -algebra \mathcal{B} or $\mathcal{A} = C^\infty(X, M_n(\mathbb{K}))$, where X is a smooth manifold and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ (or more general algebras of sections of $*$ -algebra bundles). Focusing on unitary representations of $U(\mathcal{A})$ which are boundary values of representations π of the ball semigroups

$$\text{ball}(\mathcal{A}) := \{a \in \mathcal{A} : p(a) < 1\}$$

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lead us in [BN15] to a complete reduction of this problem to the case where \mathcal{A} is a C^* -algebra and the representation is holomorphic on $\mathbf{ball}(\mathcal{A})$. This reduction is achieved by the fact that π factors through the universal map $\eta_{\mathcal{A}}: \mathcal{A} \rightarrow C^*(\mathcal{A}, p)$, where $C^*(\mathcal{A}, p)$ is the enveloping C^* -algebra of (\mathcal{A}, p) .¹

For a C^* -algebra \mathcal{A} , we write $e^{\mathcal{A}}$ for the c_0 -direct sum of the C^* -algebras $S^n(\mathcal{A}) \subseteq \mathcal{A}^{\otimes n}$, where the tensor products are constructed from the maximal C^* -cross norm. We have a holomorphic $*$ -homomorphism of semigroups

$$\Gamma: \mathbf{ball}(\mathcal{A}) \rightarrow e^{\mathcal{A}}, \quad \Gamma(a) = \sum_{n=0}^{\infty} a^{\otimes n}.$$

One of the main results of [BN15] implies that, for every bounded holomorphic $*$ -representation (π, V) of the multiplicative $*$ -semigroup $\mathbf{ball}(\mathcal{A})$, there exists a (linear) representation $\Phi: e^{\mathcal{A}} \rightarrow B(V)$ with $\Phi \circ \Gamma = \pi$. As $e^{\mathcal{A}}$ is the direct sum of the ideals $S^n(\mathcal{A})$, this in turn reduces all classification issues to representation theory of the C^* -algebras $S^n(\mathcal{A})$. Via the homogeneous multiplicative map

$$\Gamma_n: \mathcal{A} \rightarrow S^n(\mathcal{A}), \quad a \mapsto a^{\otimes n}, \quad (1)$$

the representations of $S^n(\mathcal{A})$ are in one-to-one correspondence with multiplicative holomorphic $*$ -representations of the multiplicative $*$ -semigroup (\mathcal{A}, \cdot) (or the unit ball $\mathbf{ball}(\mathcal{A})$) which are homogeneous of degree n . We call them *polynomial representations of \mathcal{A}* . As $\mathbf{U}(\mathcal{A})$ is a totally real submanifold of \mathcal{A} (resp., of $\mathbf{1} + \mathcal{A}$ in the non-unital case), these representations are uniquely determined by their restrictions to $\mathbf{U}(\mathcal{A})$, which are norm-continuous unitary representations. The multiplicative maps (1) lead to natural maps

$$S^n(\mathcal{A})^{\wedge} \xrightarrow{\Gamma_n^*} \mathbf{U}(\mathcal{A})^{\wedge}, \quad \Gamma_n^*(\pi)(a) := \pi(a^{\otimes n}),$$

which embeds the C^* -algebra dual space $S^n(\mathcal{A})^{\wedge}$ into the set $\mathbf{U}(\mathcal{A})^{\wedge}$ of equivalence classes of unitary irreducible representations of $\mathbf{U}(\mathcal{A})$. This method of constructing points in $\mathbf{U}(\mathcal{A})^{\wedge}$ requires a detailed understanding of the representation theory of the C^* -algebra $S^n(\mathcal{A})$ in terms of the representation theory of \mathcal{A} , and this is the main goal of the present paper.

Representations of unitary groups are thus related to the interaction between C^* -algebras and infinite dimensional analyticity — a theme that was developed in the comprehensive monograph [Up85] and has been explored until recently, for instance in the prequel [BN15] to the present paper. We touch on that theme here again, proving, among other things, that holomorphy, although much weaker than linearity, is strong enough to imply for instance complete positivity of multiplicative homomorphisms of C^* -algebras (see Proposition 3.2 below). The present approach to representation theory of infinite dimensional unitary groups is based on C^* -algebras and their multiplicative representations and is thus complementary to other recent works on representation theory of infinite dimensional Lie groups, such as [Wo14] or [DwOl15]. This approach allows us to shed fresh light on, and to partially extend, some results on Schur–Weyl duality in infinite dimensions from [BN12], [Nes13], and [EnIz15].

If \mathcal{A} is a unital C^* -algebra, its unitary group $\mathbf{U}(\mathcal{A})$ is a Banach–Lie group whose representation theory is much more complex than the representation theory of \mathcal{A} . That fact

¹ As we learned from Jan Stochel, he obtained closely related results in [St92] which has some overlap with [BN15].

is already apparent in the special case of matrix algebras $\mathcal{A} = M_k(\mathbb{C})$, when $U(\mathcal{A})$ is the compact Lie group $U_k(\mathbb{C})$ with a rich representation theory, while the tautological representation on \mathbb{C}^k is the only irreducible representation of $M_k(\mathbb{C})$ up to unitary equivalence. This classical situation is of course well understood using various well-known tools, including for instance

- the Peter–Weyl theorem on compact groups;
- the Cartan–Weyl Theorem on the highest weights of irreducible representations;
- the Schur–Weyl theory on tensor realizations of representations of classical groups.

These tools are quite specific to finite dimensions, inasmuch as they rely on Haar measures or on eigenvectors of the adjoint representation.

We have shown in [BN12] that, for any unital C^* -algebra \mathcal{A} , the Schur–Weyl method leads for every irreducible representation (π, \mathcal{H}) of \mathcal{A} to an infinite family $(\pi_\lambda, \mathbb{S}_\lambda(\mathcal{H}))$ of irreducible representations of $U(\mathcal{A})$, where $\lambda \in \text{Part}(n)$ is a partition of n and $\mathcal{H}^{\otimes n}$ decomposes under $U(\mathcal{A})$ as $\bigoplus_{\lambda \in \text{Part}(n)} \mathbb{S}_\lambda(\mathcal{H})^{\oplus d_\lambda}$, where d_λ is the dimension of the irreducible representation of S_n corresponding to λ . In [BN12] it was left unclear to which extent one thus exhausts the unitary dual $U(\mathcal{A})^\wedge$ of $U(\mathcal{A})$. As a by-product of our investigation, we prove in Theorem 5.8 that if \mathcal{A} is a separable C^* -algebra, then the unitary dual $U(\mathcal{A})^\wedge$ is exhausted by the representations constructed by the Schur–Weyl method if and only if \mathcal{A} is the C^* -algebra $K(\mathcal{H})$ of compact operators on a separable complex Hilbert space. Among other things, this explains why the remarkably complete results of [Ki73] on unitary irreducible representations of $U(\mathcal{A})$ for $\mathcal{A} = K(\mathcal{H})$ were never extended to unitary groups of more general C^* -algebras.

Our approach to the representation theory of $S^n(\mathcal{A})$ is based on the fact that this algebra is the fixed-point set of the permutation action of the permutation group S_n on the tensor power $\mathcal{A}^{\otimes n}$, hence we can use the old observation of [Ro79] for describing the dual space $S^n(\mathcal{A})^\wedge$ as an open subset of the dual space of the corresponding crossed product $\mathcal{A}^{\otimes n} \rtimes S_n$. We actually extend that approach to unitary equivalence classes of factor representations, because the unitary group $U(\mathcal{A})$ in general has norm-continuous unitary factor representations of type II and III. In this connection, some specific applications are given in Theorem 6.4.

The structure of this paper is as follows. In Section 2 we discuss the structure of the symmetric powers $S^n(\mathcal{A})$ of a few specific types of C^* -algebras. In Section 3 we then take a closer look at multiplicative homomorphisms, resp., representations of C^* -algebras. Here our main result is Theorem 3.2, asserting that every multiplicative homomorphism of C^* -algebras is completely positive, contractive and a sum of homogeneous homomorphisms. To understand the passage from \mathcal{A} to $S^n(\mathcal{A})$, it is natural to study $S^n(\mathcal{A})$ as the fixed point algebra $(\mathcal{A}^{\otimes n})^{S_n}$ for the natural action of the symmetric group S_n on $\mathcal{A}^{\otimes n}$. Therefore we study in Section 4 C^* -dynamical systems (\mathcal{A}, G, α) , where G is a finite group. Here our main point is to describe the relation between representations of the fixed point algebra \mathcal{A}^G and the crossed product $\mathcal{A} \rtimes_\alpha G$, in which \mathcal{A}^G is embedded as a hereditary subalgebra. Here the main results are the description of the factor representations of the crossed product in terms of induced representations (Theorem 4.14), its refinement for irreducible representations (Proposition 4.18) and, eventually, the description of $\widehat{\mathcal{A}^G}$ in Theorem 4.23. All this is applied in Section 5 to the special case $(\mathcal{A}^{\otimes n}, S_n, \alpha)$. For C^* -algebras of type I, we obtain in Theorem 5.3 a description of the equivalence classes of the irreducible representations of

$S^n(\mathcal{A})$ in terms of $\mathcal{A}^{\otimes n} \rtimes_{\alpha} S_n$ and we also relate this to the Schur–Weyl theory for C^* -algebras developed in [BN12]. In the final Section 6 we develop some aspects of Schur–Weyl theory for factor representations. This generalizes parts of recent work of Nessonov [Nes13] and Enomoto/Izumi [EnIz15, Th. 4.1] to more general von Neumann algebras. Our main results (Proposition 6.2 and Theorem 6.4) imply that the multiplicative n -fold tensor power representation on $\mathcal{H}^{\otimes n}$ for a factor $\mathcal{M} \subseteq B(\mathcal{H})$ of type II_1 , II_{∞} or III is of the same type and we even obtain some more detailed information on the corresponding projections.

Basic notation. We will use the following notation for any C^* -algebra \mathcal{A} :

- $U(\mathcal{A}) := \{u \in \mathcal{A} \mid u^*u = uu^* = \mathbf{1}\}$ for the *unitary group* of a unital C^* -algebra \mathcal{A} and for a non-unital C^* -algebra we put $U(\mathcal{A}) := U(\mathcal{A}^1) \cap (\mathbf{1} + \mathcal{A})$, where $\mathcal{A}^1 = \mathcal{A} \oplus \mathbb{C}\mathbf{1} \subseteq M(\mathcal{A})$ is the unitization of \mathcal{A} , and $M(\mathcal{A})$ is the multiplier algebra of \mathcal{A} ;
- $\text{ball}(\mathcal{A}) := \{a \in \mathcal{A} \mid \|a\| < 1\}$ for the *open ball semigroup* of \mathcal{A} ;
- $\overline{\text{ball}}(\mathcal{A}) := \{a \in \mathcal{A} \mid \|a\| \leq 1\}$ for the *closed ball semigroup* of (\mathcal{A}, p) .

Both $\text{ball}(\mathcal{A})$ and $\overline{\text{ball}}(\mathcal{A})$ are $*$ -semigroups, and if \mathcal{A} is unital, then $\mathbf{1} \in \overline{\text{ball}}(\mathcal{A}, p)$. If \mathcal{H} is a complex Hilbert space and $B(\mathcal{H})$ is the C^* -algebra of all bounded linear operators on \mathcal{H} , then we also write $C(\mathcal{H}) := \overline{\text{ball}}(B(\mathcal{H})) = \{T \in B(\mathcal{H}) \mid \|T\| \leq 1\}$.

We denote by \simeq the relation of unitary equivalence and by \approx the (weaker) relation of quasi-equivalence of group or algebra representations on Hilbert spaces [Dix64]. For any C^* -algebra \mathcal{A} we denote by $\widehat{\mathcal{A}}$ its set of unitary equivalence classes of irreducible $*$ -representations. For every irreducible $*$ -representation $\pi: \mathcal{A} \rightarrow B(\mathcal{H}_{\pi})$, we denote its unitary equivalence class by $[\pi] \in \widehat{\mathcal{A}}$. The set $\widehat{\mathcal{A}}$ is considered as a topological space endowed with the Fell topology, which may be described by the condition that the correspondence $\mathcal{J} \mapsto \{[\pi] \in \widehat{\mathcal{A}} \mid \mathcal{J} \not\subseteq \ker \pi\}$ is a bijection between the closed two-sided ideals $\mathcal{J} \subseteq \mathcal{A}$ and the open subsets of $\widehat{\mathcal{A}}$.

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2 Some examples of symmetric powers of C^* -algebras

As mentioned above, in the present paper we develop some mechanisms to get more explicit access to the C^* -algebras $S^n(\mathcal{A})$ for a given C^* -algebra \mathcal{A} . To illustrate that idea, we will briefly discuss here symmetric tensor powers of some basic examples of C^* -algebras.

Example 2.1. If X is a compact space and $\mathcal{A} = C(X)$, then, for every $n \in \mathbb{N}$, we have $\mathcal{A}^{\otimes n} \cong C(X^n)$ because $\text{Hom}(\mathcal{A}^{\otimes n}, \mathbb{C}) \cong \text{Hom}(\mathcal{A}, \mathbb{C})^n$. From that we immediately infer that

$$S^n(\mathcal{A}) \cong C(X^n/S_n)$$

where X^n/S_n stands for the quotient space with respect to the natural action of the symmetric group S_n by permutations of the coordinates of points in the n th Cartesian power X^n . Note that X^n/S_n need not be a smooth manifold if X is.

If $X = \{x_1, \dots, x_N\}$ is a finite set, then $C(X) \cong \mathbb{C}^N$, hence the above remarks imply that for any $n \geq 1$, we have a linear isomorphism

$$S^n(\mathbb{C}^N) \cong \mathbb{C}^{r(N,n)}.$$

Here $r(N, n) := |X^n/S_n| = \binom{N+n-1}{n} = \frac{(N+n-1)!}{n!(N-1)!}$ is the number of combinations of N elements taken by n at a time, where we allow any element to be selected repeatedly (equivalently, the number of non-decreasing functions $\{1, \dots, n\} \rightarrow \{1, \dots, N\}$).

Example 2.2. Let $\mathcal{A} := C(X, \mathcal{B})$ for a compact space X and a unital C^* -algebra \mathcal{B} . It is well known that $C(X)$ is nuclear and one has a canonical $*$ -isomorphism $\mathcal{A} \cong C(X) \otimes \mathcal{B}$ ([Bl06, Th. II.9.4.4], [Mu90, Thm. 6.4.17]). We want to determine the C^* -algebra $S^n(\mathcal{A})$. It is easy to see that

$$\mathcal{A}^{\otimes n} \cong (C(X) \otimes \mathcal{B})^{\otimes n} \cong C(X^n, \mathcal{B}^{\otimes n}),$$

where S_n acts on this algebra by

$$(\sigma F)(x_1, \dots, x_n) := \sigma.F(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)}).$$

Accordingly, $S^n(\mathcal{A})$ can be identified with the set $C(X^n, \mathcal{B}^{\otimes n})^{S_n}$ of S_n -equivariant maps $F: X^n \rightarrow \mathcal{B}^{\otimes n}$. In this sense it is an *equivariant map algebra* (cf. [NS13]).

The center of $\mathcal{A}^{\otimes n}$ contains $C(X^n)$, so that we obtain

$$Z(S^n(\mathcal{A})) \supseteq S^n(C(X)) \cong C(X^n)^{S_n} \cong C(X^n/S_n)$$

(see [St92, Prop. 12.2] for more details). Since every irreducible representation of the C^* -algebra $S^n(\mathcal{A})$ determines a central character $\chi: S^n(C(X)) \rightarrow \mathbb{C}$, and all these characters are given by evaluation in an S_n -orbit $\mathbf{x} := S_n \cdot (x_1, \dots, x_n) =: [x_1, \dots, x_n] \in X^n/S_n$, it factors through the restriction map given by

$$\text{ev}_{\mathbf{x}}: S^n(\mathcal{A}) \cong C(X^n, \mathcal{B}^{\otimes n})^{S_n} \rightarrow C(\mathbf{x}, \mathcal{B}^{\otimes n})^{S_n}, \quad f \mapsto f|_{\mathbf{x}}.$$

Let $S_{n,\mathbf{x}}$ be the stabilizer of some representative $(x_1, \dots, x_n) \in \mathbf{x}$, so that $\mathbf{x} \cong S_n/S_{n,\mathbf{x}}$. Evaluating in the representative, we see that the image of $\text{ev}_{\mathbf{x}}$ is isomorphic to $(\mathcal{B}^{\otimes n})^{S_{n,\mathbf{x}}}$. Assuming without loss of generality that $|\{x_1, \dots, x_n\}| = k$, we see that

$$S_{n,\mathbf{x}} \cong S_{n_1} \times \dots \times S_{n_k} \quad \text{for} \quad n_1 + \dots + n_k = n$$

is a product of symmetric groups. This further leads to

$$C(\mathbf{x}, \mathcal{B}^{\otimes n})^{S_n} \cong (\mathcal{B}^{\otimes n})^{S_{n, \mathbf{x}}} \cong \bigotimes_{j=1}^k (\mathcal{B}^{\otimes n_j})^{S_{n_j}} \cong \bigotimes_{j=1}^k S^{n_j}(\mathcal{B}).$$

Therefore the determination of the irreducible representations of $S^n(\mathcal{A})$ is reduced to the determination of irreducible representations of finite tensor products of the algebras $S^m(\mathcal{B})$. If \mathcal{B} is of type I, then so is every $S^m(\mathcal{B})$ (see Proposition 4.21(v) below). Further, irreducible representations of tensor powers of type I algebras are tensor products of irreducible representations. This reduces the problem to determine the irreducible representations of $S^n(\mathcal{A})$ to the corresponding problem for the target algebras $S^n(\mathcal{B})$.

From [BN12] we know that every irreducible representation (π, \mathcal{H}) of a C^* -algebra \mathcal{A} and every partition λ of n determine an irreducible representation $(\pi_\lambda, \mathbb{S}_\lambda(\mathcal{H}))$ of $S^n(\mathcal{A})$. One may expect to obtain from every faithful irreducible representation (π, \mathcal{H}) of \mathcal{A} a decomposition

$$S^n(\mathcal{A}) \cong \bigoplus_{\lambda \in \text{Part}(n)} S_\lambda^n(\mathcal{A}) \quad (2)$$

from the corresponding decomposition of $S^n(B(\mathcal{H}))$, acting on $\mathcal{H}^{\otimes n}$, where it commutes with the action of S_n . We shall see in Example 2.5 below that this is far from being true in general, as the example $\mathcal{A} = B(\mathcal{H})$ shows. However, it is true for the ideal $K(\mathcal{H})$ of compact operators.

Example 2.3. Let $\mathcal{A} = K(\mathcal{H})$ be the C^* -algebra of compact operators on the infinite dimensional complex Hilbert space \mathcal{H} . Using the fact that for any finite dimensional complex vector spaces \mathcal{V}_1 and \mathcal{V}_2 one has a canonical isomorphism $\text{End}(\mathcal{V}_1) \otimes \text{End}(\mathcal{V}_2) \cong \text{End}(\mathcal{V}_1 \otimes \mathcal{V}_2)$ and approximating the compact operators by finite-rank operators, we obtain, for any integer $n \geq 1$, a canonical isomorphism of C^* -algebras $\mathcal{A}^{\otimes n} \cong K(\mathcal{H}^{\otimes n})$ (see [RW98, Ex. B.20] for details). This isomorphism intertwines the permutation action of S_n on $\mathcal{A}^{\otimes n}$ and the spatial action of S_n on $K(\mathcal{H}^{\otimes n})$ defined via the unitary representation

$$\tau: S_n \rightarrow \text{U}(\mathcal{H}^{\otimes n}), \quad \tau(\sigma)(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}.$$

defined by permutation of factors in tensor monomials. Therefore, considering the fixed-point algebras of the above two actions of S_n , we see that the isomorphism $\mathcal{A}^{\otimes n} \cong K(\mathcal{H}^{\otimes n})$ defines by restriction an isomorphism

$$S^n(\mathcal{A}) \cong K(\mathcal{H}^{\otimes n}) \cap \tau(S_n)' \cong \bigoplus_{\lambda \in \text{Part}(n)} K(\mathbb{S}_\lambda(\mathcal{H})), \quad (3)$$

where $\mathbb{S}_\lambda(\mathcal{H}) = B(V_\lambda, \mathcal{H}^{\otimes n})^{S_n}$ is the multiplicity space for the irreducible representation $(\rho_\lambda, V_\lambda)$ of S_n in $\mathcal{H}^{\otimes n}$. In fact, from the S_n -decomposition

$$\mathcal{H}^{\otimes n} \cong \bigoplus_{\lambda \in \text{Part}(n)} \mathbb{S}_\lambda(\mathcal{H}) \otimes V_\lambda$$

it follows that a compact operator on $\mathcal{H}^{\otimes n}$ commutes with S_n if and only if it is a direct sum of operators of the form $A_\lambda \otimes \mathbf{1}$, where $A_\lambda \in B(\mathbb{S}_\lambda(\mathcal{H}))$. Note that (3) is the decomposition of $S^n(\mathcal{A})$ according to the general structure theory of C^* -algebras of compact operators as developed for instance in [Ar76, Thm. 1.4.5].

Example 2.4. If \mathcal{A} and \mathcal{B} are any C^* -algebras and $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ is a $*$ -homomorphism, then for any integer $n \geq 1$ one has natural $*$ -homomorphisms

$$\varphi^{\otimes n}: \mathcal{A}^{\otimes n} \rightarrow \mathcal{B}^{\otimes n} \quad \text{and} \quad S^n(\varphi): S^n(\mathcal{A}) \rightarrow S^n(\mathcal{B}),$$

and $\bullet^{\otimes n}$ and $S^n(\bullet)$ are functors from the category of C^* -algebras into itself. The behavior of these functors with respect to inductive limits of nuclear C^* -algebras sheds some light on symmetric tensor powers of AF-algebras, that is, C^* -algebras that are inductive limits of finite dimensional C^* -algebras.

In fact, let \mathcal{A} be any nuclear C^* -algebra with a sequence of nuclear closed $*$ -subalgebras $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \dots \subseteq \mathcal{A}$ whose union is dense in \mathcal{A} . It is easily seen that one has a sequence of closed $*$ -subalgebras $\mathcal{A}_1^{\otimes n} \subseteq \mathcal{A}_2^{\otimes n} \subseteq \dots \subseteq \mathcal{A}^{\otimes n}$ whose union is dense in $\mathcal{A}^{\otimes n}$, and then a sequence of closed $*$ -subalgebras $S^n(\mathcal{A}_1) \subseteq S^n(\mathcal{A}_2) \subseteq \dots \subseteq S^n(\mathcal{A})$ whose union is dense in $S^n(\mathcal{A})$. Thus each of the functors $\bullet^{\otimes n}$ and $S^n(\bullet)$ commutes with inductive limits of nuclear C^* -algebras. Since these functors preserve the category of finite dimensional C^* -algebras, it then follows that they also preserve the category of AF-algebras.

We point out however that, as a by-product of Example 2.3, if \mathcal{A} is a UHF-algebra (that is, a unital C^* -algebra that is the inductive limit of a sequence of unital $*$ -subalgebras isomorphic to full matrix algebras) and $n \geq 1$, then $S^n(\mathcal{A})$ is not a UHF-algebra. More precisely, Example 2.3 shows that if $\mathcal{A} = M_k(\mathbb{C})$ for some $k \geq 2$, then $S^n(\mathcal{A})$ is the direct sum of at least two full matrix algebras if $n \geq 2$.

Now let $\mathcal{A} \cong \bigotimes_{k=1}^{\infty} B(E_k)$ with $E_k \cong \mathbb{C}^{n_k}$ be an UHF algebra. Since \mathcal{A} is a direct limit of finite dimensional simple matrix algebras $\mathcal{A}_N \cong B(F_N)$ with $F_N \cong E_1 \otimes \dots \otimes E_N$, the C^* -algebra $S^n(\mathcal{A})$ is the direct limit of the C^* -algebras $S^n(\mathcal{A}_N) \cong \bigoplus_{\lambda \in \text{Part}(n)} S^n(\mathcal{A}_N)_\lambda$ with $S^n(\mathcal{A}_N)_\lambda \cong B(\mathbb{S}_\lambda(F_N))$ (Example 2.3), where the projection onto the λ -summand corresponds to restriction to the corresponding S_n -isotypic subspace in $F_N^{\otimes n}$. From the S_n -equivariance of the inclusions $F_N^{\otimes n} \hookrightarrow F_{N+1}^{\otimes n}$, it follows that the inclusion $\mathcal{A}_N \hookrightarrow \mathcal{A}_{N+1}$ induces inclusions $S^n(\mathcal{A}_N)_\lambda \hookrightarrow S^n(\mathcal{A}_{N+1})_\lambda$, so that we obtain a direct sum decomposition

$$S^n(\mathcal{A}) \cong \bigoplus_{\lambda \in \text{Part}(n)} S^n(\mathcal{A})_\lambda \quad \text{with} \quad S^n(\mathcal{A})_\lambda \cong \varinjlim S^n(\mathcal{A}_N)_\lambda,$$

and the construction implies in particular that the algebras $S^n(\mathcal{A})_\lambda$ are simple and act irreducibly on $\mathbb{S}_\lambda(F)$, where F is the Hilbert direct limit of the spaces F_N . Therefore we find a decomposition similar to the one in Example 2.3.

Example 2.5. Let \mathcal{H} be an infinite dimensional Hilbert space.

(a) *The representation of $S^2(B(\mathcal{H}))$ on $\mathcal{H}^{\otimes 2}$ is not injective.* First we recall the non-uniqueness of C^* -norms on $B(\mathcal{H}) \otimes B(\mathcal{H})$ from [JP95] which implies that the spatial tensor product (corresponding to the least C^* -norm by [Mu90, Thm. 6.4.18]), which is the image $B(\mathcal{H})^{\bar{\otimes} 2}$ of $B(\mathcal{H})^{\otimes 2}$ under the natural representation on $\mathcal{H}^{\otimes 2}$ differs from the tensor product $B(\mathcal{H})^{\otimes 2}$ with respect to the maximal C^* -norm. This implies that the kernel $\mathcal{J} \subseteq B(\mathcal{H})^{\otimes 2}$ of the representation on $\mathcal{H}^{\otimes 2}$ is non-zero. Clearly, \mathcal{J} is invariant under the flip automorphism $\tau(a_1 \otimes a_2) = a_2 \otimes a_1$. If $0 \leq a \in \mathcal{J}$ is non-zero, then the same holds for $a + \tau(a) \in \mathcal{J} \cap S^2(B(\mathcal{H}))$. Therefore the representation of $S^2(B(\mathcal{H}))$ on $\mathcal{H}^{\otimes 2}$ is also not injective.

(b) *The representation of $S^2(B(\mathcal{H}))$ on $\mathcal{H}^{\otimes 2}$ is not surjective onto $B(\mathcal{H}^{\otimes 2})^{S_2}$.* Let $(e_j)_{j \in J}$ be an orthonormal basis of \mathcal{H} and $P \in B(\mathcal{H}^{\otimes 2})$ be the orthogonal projection onto the subspace

spanned by $(e_j \otimes e_j)_{j \in J}$. Then P commutes with the flip action of S_2 and it remains to show that it is not contained in the spatial tensor product $B(\mathcal{H})^{\bar{\otimes} 2}$. To this end, we write $\mathcal{H}^{\bar{\otimes} 2} \cong \bigoplus_{j \in J} (\mathcal{H}_j \otimes e_j)$ as a direct sum and, accordingly, P as a matrix $(P_{ij})_{i,j \in J}$. Then $P_{ij}(v) = \delta_{ij} \langle v, e_j \rangle e_j$ is either 0 or the orthogonal projection onto $\mathbb{C}(e_j \otimes e_j)$. In particular, the subset $\{P_{ij} : i, j \in J\} \subseteq B(\mathcal{H})$ is closed, discrete and infinite, hence not relatively compact. According to Exercise 11.5.7 in [KR92], this implies that P is not contained in $B(\mathcal{H})^{\bar{\otimes} 2}$.

3 Multiplicative homomorphisms of C^* -algebras

In this section we take a closer look at multiplicative homomorphisms, resp., representations of C^* -algebras. Here our main result is Theorem 3.2, asserting that every multiplicative homomorphism of C^* -algebras is completely positive, contractive and a sum of homogeneous homomorphisms. We shall use some results and notions from our earlier paper [BN15] on completely positive nonlinear maps and their dilation theory.

Definition 3.1. If \mathcal{A} and \mathcal{B} are any unital C^* -algebras, then a map $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ is a *multiplicative homomorphism* if it is holomorphic and satisfies

$$\varphi(\mathbf{1}) = \mathbf{1}, \quad \varphi(ab) = \varphi(a)\varphi(b) \quad \text{and} \quad \varphi(a^*) = \varphi(a)^* \quad \text{for} \quad a, b \in \mathcal{A}.$$

A linear map $\psi : \mathcal{A} \rightarrow \mathcal{B}$ is called *contractive* if it satisfies $\psi(\overline{\text{ball}}(\mathcal{A})) \subseteq \overline{\text{ball}}(\mathcal{B})$.

Theorem 3.2. *Every multiplicative homomorphism of unital C^* -algebras $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ is completely positive, contractive and has a weakly convergent series expansion $\varphi = \sum_{n \geq 0} \varphi_n$, where φ_n is multiplicative and homogeneous of degree n .*

The proof of the above result is based on the following observations. For the case of a positively homogeneous map φ , the complete positivity of φ can also be derived from [St92, Prop. 11.1].

Lemma 3.3. *Every dilatable bounded completely positive function $\varphi : \text{ball}(\mathcal{A}) \rightarrow B(V)$ extends to a weakly continuous completely positive function $\bar{\varphi} : \overline{\text{ball}}(\mathcal{A}) \rightarrow B(V)$.*

Proof. First recall from [BN15] that $S^\circ := \text{ball}(\mathcal{A})$ is a semigroup ideal in $S := \overline{\text{ball}}(\mathcal{A})$. Let $(\pi, \mathcal{H}, \iota) = (\pi_\varphi, \mathcal{H}_\varphi, \iota_\varphi)$ be the minimal dilation of φ given by [BN15, Prop. 3.4]. Then we use [BN15, Prop. 3.21] to extend the representation π to a weakly continuous representation $\hat{\pi}$ of S , where the weak continuity follows by the explicit formula of $\hat{\pi}$, since φ is norm-continuous by [BN15, Th. 6.4]. Note that the representation π is completely positive by [BN15, Prop. 4.1], hence, by continuity, also $\hat{\pi}$ is completely positive. Then $\bar{\varphi}(s) := \iota^* \hat{\pi}(s) \iota$ is a completely positive weakly continuous extension of φ to S . \square

Proposition 3.4. *Let \mathcal{A} be any unital C^* -algebra, and $\pi : \text{ball}(\mathcal{A}) \rightarrow B(\mathcal{H}_\pi)$ be any holomorphic $*$ -representation. Then π is completely positive, and extends to a weakly continuous $*$ -representation $\bar{\pi} : \overline{\text{ball}}(\mathcal{A}) \rightarrow C(\mathcal{H}_\pi)$. If π is nondegenerate, then $\lim_{r \rightarrow 1^-} \pi(r\mathbf{1}) = \mathbf{1}$ in the weak operator topology.*

Proof. The map π is positive definite because it is a homomorphism of $*$ -semigroups. Therefore, after we will have proved that π is contractive, hence in particular bounded, it will follow by [BN15, Th. 1.1] that it is completely positive. Then Lemma 3.3 ensures that π extends to a weakly continuous $*$ -representation of $\mathbf{ball}(\mathcal{A})$.

The proof of the fact that π is contractive has two stages.

Step 1: We first consider the case $\mathcal{A} = \mathbb{C}$, hence $\mathbf{ball}(\mathcal{A}) = \mathbb{D}$ is the unit disk. Then there exists the norm-convergent power series expansion $\pi(z) = \sum_{n \geq 0} z^n P_n \in B(\mathcal{H}_\pi)$ for all $z \in \mathbb{D}$, with $P_n \in B(\mathcal{H}_\pi)$ for all $n \geq 0$. For all $z, w \in \mathbb{D}$ one has $\pi(\bar{z}) = \pi(z)^*$ and $\pi(zw) = \pi(z)\pi(w)$, and this implies that the coefficients of the above power series satisfy the conditions $P_n = P_n^* = P_n^2$ and $P_n P_m = 0$ for all $n, m \geq 0$ with $n \neq m$. That is, $(P_n)_{n \geq 0}$ is a sequence of mutually orthogonal projections, and then

$$(\forall z \in \mathbb{D}) \quad \|\pi(z)\| \leq \sup_{n \geq 0} |z^n| = 1$$

which in particular shows that π is bounded. Moreover, one has $\lim_{0 < r \rightarrow 1^-} \pi(r\mathbf{1}) = \sum_{n \geq 0} P_n$ in the weak operator topology in $B(\mathcal{H}_\pi)$.

Step 2: In the general case, for all $z \in \mathbb{D}$ and $a \in \mathbf{ball}(\mathcal{A})$ one has $\pi(za) = \pi(z\mathbf{1})\pi(a)$, hence if $\pi \not\equiv 0$, then necessarily $\pi|_{\mathbb{D}} \not\equiv 0$, which we may assume until the end of this proof. Then use the above stage to find the sequence of mutually orthogonal projections $(P_n)_{n \geq 0}$ in $B(\mathcal{H}_\pi)$ with $\pi(z\mathbf{1}) = \sum_{n \geq 0} z^n P_n \in B(\mathcal{H}_\pi)$ for all $z \in \mathbb{D}$.

For each $a \in \mathbf{ball}(\mathcal{A})$ and all $z \in \mathbb{D}$ one has $\pi(z\mathbf{1})\pi(a) = \pi(a)\pi(z\mathbf{1})$, and this implies $P_n \pi(a) = \pi(a) P_n$, hence $\pi_n(\cdot) := P_n \pi(\cdot)$ is a holomorphic $*$ -representation of $\mathbf{ball}(\mathcal{A})$. Moreover $\pi_n(za) = z^n \pi_n(a)$ for all $a \in \mathbf{ball}(\mathcal{A})$ and $z \in \mathbb{D}$, hence one may use [BN15, Lemma 2.11] to extend π_n to a holomorphic $*$ -representation $\pi_n: (\mathcal{A}, \cdot) \rightarrow B(\mathcal{H}_\pi)$ which is homogeneous of degree n . Since π_n is continuous at $0 \in \mathcal{A}$, there exist $r, M > 0$ with $\|\pi_n(a)\| \leq M$ if $a \in \mathcal{A}$ with $\|a\| \leq r$. Since π_n is homogeneous, it then follows that it is bounded on $\mathbf{ball}(\mathcal{A})$, and then [BN15, Rem. 3.5] implies $\|\pi_n(a)\| \leq 1$ for all $a \in \mathbf{ball}(\mathcal{A})$.

Denoting $P := \sum_{n \geq 0} P_n$, we then obtain the weakly convergent series

$$\sum_{n \geq 0} \pi_n(a) = P\pi(a) = \pi(a) \quad \text{and} \quad \|\pi(a)\| \leq \sup_{n \geq 0} \|\pi_n(a)\| \leq 1$$

for every $a \in \mathbf{ball}(\mathcal{A})$. The condition that π be nondegenerate means $P = \mathbf{1}$, and the assertion follows. \square

Finally, we can now prove Theorem 3.2.

Proof. Using a realization of \mathcal{B} as an operator algebra, we may actually assume $\mathcal{B} = B(\mathcal{H})$ for some complex Hilbert space \mathcal{H} . Then Proposition 3.4 and its proof show that the restricted map $\pi := \varphi|_{\mathbf{ball}(\mathcal{A})}: \mathbf{ball}(\mathcal{A}) \rightarrow B(\mathcal{H})$ is completely positive and there exists a sequence of mutually orthogonal projections $(P_n)_{n \geq 0}$ such that $\pi_n := \pi(\cdot)P_n = P_n\pi(\cdot)$ is homogeneous of degree n and completely positive on $\mathbf{ball}(\mathcal{A})$, hence in fact $\pi_n: \mathcal{A} \rightarrow B(\mathcal{H})$ is completely positive.

On the other hand, since $(P_n)_{n \geq 0}$ are mutually orthogonal projections, we have $\varphi = \sum_{n \geq 0} \pi_n$, this series being convergent in the pointwise weak operator topology. Therefore also φ is completely positive. \square

Remark 3.5. In connection with Theorem 3.2 we point out that for some simple holomorphic maps on a C^* -algebra \mathcal{A} the complete positivity property is a condition that has strong structural implications on \mathcal{A} . For instance, if \mathcal{A} is unital and we define $\varphi: \mathcal{A} \rightarrow \mathcal{A}$, $\varphi(a) := a^2$, then φ is a holomorphic positive contractive map, and the following conditions are equivalent:

- (i) The map φ is completely positive.
- (ii) The map φ is a multiplicative homomorphism.
- (iii) The C^* -algebra \mathcal{A} is commutative.

In fact, one has (i) \Leftrightarrow (iii) for instance by [JiT03, Th. 3], and it is clear that (iii) \Rightarrow (ii). Conversely, if we assume that φ is a multiplicative homomorphism, then for arbitrary $a, b \in \mathcal{A}$ and $t, s \in \mathbb{R}$ one has $(e^{ta}e^{sb})^2 = \varphi(e^{ta}e^{sb}) = \varphi(e^{ta})\varphi(e^{sb}) = (e^{ta})^2(e^{sb})^2$, which is equivalent to $e^{sb}e^{ta} = e^{ta}e^{sb}$. Differentiating this equality with respect to s and t at $s = t = 0$, we obtain $ba = ab$, hence the C^* -algebra \mathcal{A} is commutative.

4 Representations of crossed products by finite groups

In this section, unless otherwise mentioned, (\mathcal{A}, G, α) is a C^* -dynamical system, where G is a finite group that acts by $*$ -automorphisms of a C^* -algebra \mathcal{A} by $(g, x) \mapsto \alpha_g(x)$. We investigate here how the covariant representations of these data are related to the representations of the fixed-point subalgebra

$$\mathcal{A}^G := \{x \in \mathcal{A} \mid (\forall g \in G) \alpha_g(x) = x\},$$

which by [Ro79] is $*$ -isomorphic to a hereditary subalgebra of the crossed product $\mathcal{A} \rtimes_\alpha G$. These results will be later used for maximal C^* -tensor powers $\mathcal{A}^{\otimes m}$ acted on by the permutation group $G = S_m$, with the aim of describing irreducible or factor representations of the symmetric C^* -tensor powers $(\mathcal{A}^{\otimes m})^G = S^m(\mathcal{A})$, for any C^* -algebra \mathcal{A} .

The *crossed product* corresponding to the above C^* -dynamical system is the C^* -algebra $\mathcal{A} \rtimes_\alpha G := \ell^1(G, \mathcal{A}; \alpha) := \{f \mid f: G \rightarrow \mathcal{A}\}$ with the multiplication

$$(f_1 f_2)(g) := \frac{1}{|G|} \sum_{h \in G} f_1(h) \alpha_h(f_2(h^{-1}g))$$

and the involution $f^*(g) := \alpha_g(f(g^{-1}))^*$ for all $f, f_1, f_2 \in \ell^1(G, \mathcal{A}; \alpha)$ and $g \in G$. A *covariant representation* is a pair (π, U) , where $\pi: \mathcal{A} \rightarrow B(\mathcal{H})$ is a $*$ -representation and $U: G \rightarrow B(\mathcal{H})$ is a unitary representation satisfying $\pi(\alpha_g(x)) = U_g \pi(x) U_g^*$ for all $x \in \mathcal{A}$ and $g \in G$. We also define the $*$ -representation

$$\pi \rtimes U: \mathcal{A} \rtimes_\alpha G \rightarrow B(\mathcal{H}), \quad (\pi \rtimes U)(f) = \sum_{g \in G} \pi(f(g)) U_g.$$

We say that the covariant representation (π, U) is a *factor representation* if the von Neumann algebra generated by $(\pi \rtimes U)(\mathcal{A} \rtimes_\alpha G)$, that is $(\pi(\mathcal{A}) \cup \pi(G))''$, is a factor etc.

The following definition is a specialization of [Tak67, Def. 3.2] to finite groups, except that we use here the left regular representation of groups.

Definition 4.1. (induced representations) Let (\mathcal{A}, G, α) be as above. Assume that G_0 is a subgroup of the finite group G , denote by $(\mathcal{A}, G_0, \alpha_0)$ the corresponding C^* -dynamical system with $\alpha_0 := \alpha|_{G_0}$.

Then for any covariant representation (π_0, V) of $(\mathcal{A}, G_0, \alpha_0)$ on some Hilbert space \mathcal{H}_0 , its corresponding *induced representation* is the covariant representation (π, U) of (\mathcal{A}, G, α) on the Hilbert space \mathcal{H} defined as follows:

- $\mathcal{H} := \{f: G \rightarrow \mathcal{H}_0 \mid f(gh) = V_h^* f(g) \text{ if } g \in G, h \in G_0\}$;
- $U: G \rightarrow B(\mathcal{H}), (U_g f)(\tilde{g}) := f(g^{-1}\tilde{g})$;
- $\pi: \mathcal{A} \rightarrow B(\mathcal{H}), (\pi(x)f)(g) := \pi_0(\alpha_g^{-1}(x))f(g)$ (multiplication operators).

It is useful to reformulate the above definition as follows, in the spirit of [Se77, Sect. 3.3].

Remark 4.2. Let $U: G \rightarrow B(\mathcal{H})$ be a unitary representation of a finite group. Assume that a subgroup $G_0 \subseteq G$ and a closed linear subspace $\mathcal{H}_0 \subseteq \mathcal{H}$ satisfy $U_{G_0}\mathcal{H}_0 \subseteq \mathcal{H}_0$, and define $V: G_0 \rightarrow B(\mathcal{H}_0), V_h := U_h|_{\mathcal{H}_0}$.

The group representation U is unitarily equivalent to the representation induced from V if for some (and hence for every) system $\mathbf{1} = g_0, g_1, \dots, g_r$ of representatives of the left G_0 -cosets in G we have an orthogonal direct sum decomposition

$$\mathcal{H} = \bigoplus_{j=0}^r U_{g_j} \mathcal{H}_0. \quad (4)$$

If, moreover, (\mathcal{A}, G, α) is a C^* -dynamical system and one has a covariant representation (π_0, V) of $(\mathcal{A}, G_0, \alpha_0)$ on \mathcal{H}_0 , where $\alpha_0 := \alpha|_{G_0}$ and $V: G_0 \rightarrow B(\mathcal{H}_0)$ is as above, then the corresponding induced covariant representation is unitarily equivalent to the induced covariant representation (π, U) of (\mathcal{A}, G, α) on \mathcal{H} with $\pi: \mathcal{A} \rightarrow B(\mathcal{H})$ given by

$$\pi(a)|_{U_{g_j}\mathcal{H}_0} := U_{g_j}\pi_0(a)U_{g_j}^{-1}|_{U_{g_j}\mathcal{H}_0} \quad \text{for } j = 0, 1, \dots, r, x \in \mathcal{A}.$$

Remark 4.3. For later use, we note that in Definition 4.1 there exists a natural unitary operator between the spaces of fixed points

$$\mathcal{H}_0^{G_0} \rightarrow \mathcal{H}^G, \quad v \mapsto f_v$$

where $f_v: G \rightarrow \mathcal{H}_0, f_v(g) := |G/G_0|^{-1/2}v$ for all $g \in G$ and $v \in \mathcal{H}_0^{G_0}$.

Remark 4.4. In the setting of Remark 4.2, let $p_0: \mathcal{H} \rightarrow \mathcal{H}$ denote the orthogonal projection onto \mathcal{H}_0 . Then

$$G_0 = \{g \in G \mid U_g p_0 U_g^{-1} = p_0\} \subseteq \{g \in G \mid \pi_0 \circ \alpha_g \simeq \pi_0\} \subseteq \{g \in G \mid \pi_0 \circ \alpha_g \approx \pi_0\}$$

recalling that \approx stands for quasi-equivalence of $*$ -representations.

4.1 Commutants of induced representations

The situation when both inclusions in Remark 4.4 are equalities should be thought of as a maximality property of the subgroup G_0 in the same way as the polarizations at points of duals of finite dimensional Lie algebras are maximal isotropic subalgebras, which leads to irreducibility of the corresponding induced representations. That idea will be made precise in Corollary 4.6 below. To that end we need the following variant of [Tak67, Th. 4.3] for finite groups acting on C^* -algebras that need not be separable.

Proposition 4.5. *Let (π, U) be the covariant representation of (\mathcal{A}, G, α) on \mathcal{H} induced by the covariant representation (π_0, V) of $(\mathcal{A}, G_0, \alpha|_{G_0})$ on \mathcal{H}_0 . Pick any complete system $\mathbf{1} = g_0, g_1, \dots, g_r$ of representatives of the left cosets of G_0 in G and for $j = 0, \dots, r$ let $p_j \in B(\mathcal{H})$ be the orthogonal projection on $\mathcal{H}_j := U_{g_j} \mathcal{H}_0$ and $G_j := g_j G_0 g_j^{-1}$.*

Then $G_j = \{g \in G \mid U_g p_j U_g^{-1} = p_j\}$, and if we define (π_j, V_j) as the covariant representation of $(\mathcal{A}, G_j, \alpha|_{G_j})$ on \mathcal{H}_j , given by $\pi_j(\cdot) := \pi(\cdot)|_{\mathcal{H}_j}$ and $V_j(\cdot) := U(\cdot)|_{\mathcal{H}_j}$, then (π, U) is induced by (π_j, V_j) and the map

$$\Phi_j : (\pi \rtimes U)(\mathcal{A} \rtimes G)' \cap \{p_0, p_1, \dots, p_r\}' \rightarrow (\pi_j \rtimes V_j)(\mathcal{A} \rtimes G_j)', \quad T \mapsto T|_{\mathcal{H}_j}$$

is well defined and is a $$ -isomorphism of von Neumann algebras.*

Proof. The action of G permutes the subspaces $\mathcal{H}_0, \dots, \mathcal{H}_r$, and this implies that (π, U) is induced by (π_j, V_j) for $j = 0, \dots, r$.

For every $T \in (\pi \rtimes U)(\mathcal{A} \rtimes G)' \cap \{p_0, p_1, \dots, p_r\}'$ we have $T p_j = p_j T$. In particular, such an operator T is given by a block diagonal matrix with respect to the decomposition $p_0 + \dots + p_r = \mathbf{1}$, and this also shows that for any $j \in \{0, \dots, r\}$ the map Φ_j is a $*$ -homomorphism.

To check that Φ_j is injective, recall that the group G acts transitively on $\{\mathcal{H}_0, \dots, \mathcal{H}_r\}$. Therefore the subspace \mathcal{H}_j of \mathcal{H} is G -cyclic, hence separates the commutant U_G' which contains $(\pi \rtimes U)(\mathcal{A} \rtimes G)'$.

To check that the image of Φ_j is $(\pi_j \rtimes V_j)(\mathcal{A} \rtimes G_j)'$, let $T_j \in (\pi_j \rtimes V_j)(\mathcal{A} \rtimes G_j)'$. For any $g \in G$ with $U_g \mathcal{H}_j = \mathcal{H}_k$, we then obtain an operator

$$T_k := U_g T_j U_g^* : \mathcal{H}_k \rightarrow \mathcal{H}_k$$

which does not depend on the choice of g in the G_j -coset $\{x \in G : U_x \mathcal{H}_j = \mathcal{H}_k\}$. We thus obtain a block-diagonal operator $T \in B(\mathcal{H})$ with $T|_{\mathcal{H}_k} := T_k$ for $k = 0, \dots, r$. It is straightforward to verify that T commutes with $(\pi \rtimes U)(\mathcal{A} \rtimes G)$. Clearly also $T \in \{p_0, p_1, \dots, p_r\}'$ and $\Phi_j(T) = T_j$. \square

Corollary 4.6. *Suppose that, in the setting of Proposition 4.5, the following additional conditions are satisfied:*

- $G_0 = \{g \in G \mid \pi_0 \circ \alpha_g \approx \pi_0\}$;
- $\pi_0 : \mathcal{A} \rightarrow B(\mathcal{H}_0)$ is a factor representation.

Then the map

$$\Phi_j : (\pi \rtimes U)(\mathcal{A} \rtimes G)' \rightarrow (\pi_j \rtimes U_j')(\mathcal{A} \rtimes G_j)', \quad T \mapsto p_j T|_{\mathcal{H}_j}$$

is a well-defined $$ -isomorphism of von Neumann algebras for $j = 0, \dots, r$.*

Proof. Using Proposition 4.5, it suffices to show that $(\pi \rtimes U)(\mathcal{A} \rtimes G)' \subseteq \{p_0, p_1, \dots, p_r\}'$. One has $(\pi \rtimes U)(\mathcal{A} \rtimes G)' = (\pi(\mathcal{A}) \cup \pi(G))' = \pi(\mathcal{A})' \cap \pi(G)' \subseteq \pi(\mathcal{A})'$ and we will check that for arbitrary $T \in \pi(\mathcal{A})'$ one has $Tp_j = p_j T$ for $j = 0, \dots, r$.

In fact, writing the operators on \mathcal{H} as block matrices with respect to the decomposition $p_0 + \dots + p_r = \mathbf{1}$, one has $T = (T_{jk})_{0 \leq j, k \leq r}$ and $\pi(x) = \text{diag}(\pi_0(x), \pi_1(x), \dots, \pi_r(x))$ is a block diagonal matrix for all $x \in \mathcal{A}$. Hence the condition $T \in \pi(\mathcal{A})'$ is equivalent to the fact that for all $j, k = 0, \dots, r$ one has

$$(\forall x \in \mathcal{A}) \quad \pi_j(x)T_{jk} = T_{jk}\pi_k(x).$$

On the other hand the hypotheses imply that if $j \neq k$ then $\pi_j|_{\mathcal{A}}$ and $\pi_k|_{\mathcal{A}}$ are factor representations that are not quasi-equivalent, hence it follows by [Dix64, Cor. 5.3.6] that there exists no non-zero intertwiner between them, and then $T_{jk} = 0$. Therefore T is given by a block-diagonal matrix, that is, $Tp_j = p_j T$ for $j = 0, \dots, r$, and this completes the proof. \square

Remark 4.7. In the setting of Corollary 4.6, it follows that the induced covariant representation (π, U) is irreducible or is a factor representation of some type I, II, or III, if and only if so is the input representation (π_0, V) .

Remark 4.8. In the proof of Proposition 4.5, the idea that $T = \Phi_0(T) + \dots + \Phi_r(T)$ is a sum of mutually unitarily equivalent operators is related to [Co75, Lemma 3].

On the other hand, since the action of the group G on the set $\{p_0, \dots, p_r\}$ is transitive, and the isotropy groups G_0, \dots, G_r are mapped to each other by suitable inner automorphisms of G , this agrees with the uniqueness assertion on G_0 in [Tak67, Th. 5.2].

4.2 Covariant factor representations as induced representations

We now turn to a question that is somehow converse to what we did so far, namely the question of when a given covariant representation is an induced representation (cf. Proposition 4.12.) We will need the following observation which is essentially a byproduct of the proof of [AL10, Th. 3.1]. We recall that the action of G on \mathcal{A} is called *ergodic* if $\mathcal{A}^G = \mathbb{C}\mathbf{1}$.

Lemma 4.9. *If the action of the finite group G on \mathcal{A} is ergodic, then $\dim \mathcal{A} \leq |G| < \infty$.*

Proof. Define

$$\alpha: \ell^1(G) \rightarrow B(\mathcal{A}), \quad \alpha(\varphi) = \frac{1}{|G|} \sum_{g \in G} \varphi(g) \alpha_g.$$

Let π_1, \dots, π_n be a system of representatives of all equivalence classes of irreducible representations of G . For $j = 1, \dots, n$ denote by d_j the dimension of π_j , by $\chi_j: G \rightarrow \mathbb{C}$ the character of π_j , and define $\mathcal{A}_j := \alpha(\chi_j)\mathcal{A} \subseteq \mathcal{A}$, so that $\mathcal{A} = \mathcal{A}_1 \oplus \dots \oplus \mathcal{A}_n$. On the other hand, since G is in particular a compact group and its action on \mathcal{A} is ergodic by hypothesis, it follows by [HLS81, Prop. 2.1] that $\dim \mathcal{A}_j \leq d_j^2$ for $j = 1, \dots, n$. Therefore $\dim \mathcal{A} \leq \sum_{j=1}^n d_j^2 = |G|$, by Burnside's Theorem, and this concludes the proof. \square

Lemma 4.10. *Let (π, U) be a covariant representation of (\mathcal{A}, G, α) on \mathcal{H} and consider the action of G on $B(\mathcal{H})$ by $(g, T) \mapsto U_g T U_g^{-1}$.*

- (i) If (π, U) is a factor representation, then G acts ergodically on the abelian von Neumann algebra $\mathcal{Z} := \pi(\mathcal{A})'' \cap \pi(\mathcal{A})'$ and $\dim \mathcal{Z} \leq |G| < \infty$.
- (ii) If (π, U) is irreducible, then G acts ergodically on $\pi(\mathcal{A})'$ and $\dim \pi(\mathcal{A})' \leq |G| < \infty$.

Proof. If (π, U) is a factor representation, then one has

$$\begin{aligned} \mathcal{Z}^G &= \pi(\mathcal{A})'' \cap \pi(\mathcal{A})' \cap U'_G = \pi(\mathcal{A})'' \cap (\pi(\mathcal{A}) \cup U_G)' \\ &\subseteq (\pi(\mathcal{A}) \cup U_G)'' \cap (\pi(\mathcal{A}) \cup U_G)' = \mathbb{C}\mathbf{1} \end{aligned}$$

where the latter equality follows by the hypothesis that (π, U) is a factor representation. Thus $\mathcal{Z}^G = \mathbb{C}\mathbf{1}$, and then $\dim \mathcal{Z} \leq |G|$ by Lemma 4.9.

If the representation π is irreducible, then $\pi(\mathcal{A})' \cap U'_G = \mathbb{C}\mathbf{1}$, hence the action of G on $\pi(\mathcal{A})'$ is ergodic, and then $\dim \pi(\mathcal{A})' \leq |G|$ by Lemma 4.9 again. \square

We now give a basic result on $*$ -representations with finite dimensional commutant, for later use along with Lemma 4.10.

Lemma 4.11. *Let S be any $*$ -semigroup with a $*$ -representation $\pi: S \rightarrow B(\mathcal{H})$. If the center of $\pi(S)'$ is finite dimensional, then there exist a finite orthogonal direct sum decomposition $\mathcal{H} = \bigoplus_{j=1}^n \mathcal{H}_j$ with the following properties:*

- (i) *The orthogonal projections of \mathcal{H} onto $\mathcal{H}_1, \dots, \mathcal{H}_n$, respectively, are the minimal central projections in $\pi(S)'$.*
- (ii) *The representations $\pi_j := \pi(\cdot)|_{\mathcal{H}_j}: S \rightarrow B(\mathcal{H}_j)$ for $j = 1, \dots, n$, are pairwise non-quasi-equivalent factor representations.*
- (iii) *If $\pi(S)'$ is also finite dimensional, then, for $j = 1, \dots, n$, there exist mutually inequivalent irreducible $*$ -representations (ρ_j, V_j) of S and $k_j \in \mathbb{N}_0$ such that $\pi_j \cong \rho_j^{\oplus k_j}$.*

Proof. Let $\mathcal{M} := \pi(S)'$ and $p_1, \dots, p_n \in \mathcal{M}$ be the minimal central projections. We show that the assertions hold with $\mathcal{H}_j := p_j(\mathcal{H})$ for $j = 1, \dots, n$.

For Assertion (ii), let $i, j \in \{1, 2, \dots, n\}$ with $i \neq j$. Since π_i and π_j are factor representations, it follows just as in [Dix64, Prop. 5.2.9] that it suffices to prove that π_i and π_j are disjoint $*$ -representations, that is, their only intertwiner is zero. Since the set of intertwiners corresponds to $p_i \mathcal{M} p_j$, this follows from $p_i p_j = 0$.

For Assertion (iii), if \mathcal{M} is finite dimensional von Neumann algebra, then the ideals $p_j \mathcal{M}$ are isomorphic to matrix algebras $M_{k_j}(\mathbb{C})$. Accordingly, the factor representation π_j is type I and a k_j -fold multiple of an irreducible representation (ρ_j, V_j) . \square

The following result is essentially a special case of [Tak67, Th. 5.1–5.2]. We note that the finite dimensionality of \mathcal{Z} follows from Lemma 4.10(i).

Lemma 4.12. *Let (π, U) be a covariant factor representation of (\mathcal{A}, G, α) on \mathcal{H} and $\mathcal{Z} := \pi(\mathcal{A})'' \cap \pi(\mathcal{A})'$. Then the following assertions hold:*

- (i) Let $p \in \mathcal{Z}$ be a minimal projection and $\mathcal{K} := p\mathcal{H}$, $H := \{g \in G \mid U_g p U_g^{-1} = p\}$, and let (ρ, V) be the covariant representation of $(\mathcal{A}, H, \alpha|_H)$ on \mathcal{K} given by $\rho(a) := \pi(a)|_{\mathcal{K}}$ and $V_h := U_h|_{\mathcal{K}}$. Then $H = \{g \in G \mid \rho \circ \alpha_g \approx \rho\}$, ρ is a factor $*$ -representation and (π, U) is unitarily equivalent to the covariant representation induced by (ρ, V) .
- (ii) The map $\Phi: (\pi \rtimes U)(\mathcal{A} \rtimes G)' \rightarrow (\rho \rtimes V)(\mathcal{A} \rtimes H)'$, $T \mapsto pT|_{\mathcal{K}}$, is well defined and is a $*$ -isomorphism of von Neumann algebras.

Proof. For (i), we note that Lemma 4.11(ii) implies that ρ is a factor representation since \mathcal{Z} is the center of $\pi(\mathcal{A})'$. Moreover, for every $g \in G$, one has

$$(\pi \circ \alpha_g)(\cdot) = U_g \pi(\cdot) U_g^{-1},$$

so the operators U_g act by conjugation on the von Neumann algebra $\pi(\mathcal{A})''$, hence also on its center \mathcal{Z} . The action of G on \mathcal{Z} is ergodic by Lemma 4.10(i), and this implies that the action of G on the set $\{p_1, \dots, p_n\}$ of minimal idempotents in \mathcal{Z} is transitive (cf. [Ka13b, Lemma 3.4]).

Accordingly, G permutes the subspaces $\mathcal{H}_j := p_j \mathcal{H}$ transitively, and this implies by Remark 4.2 that (π, U) is unitarily equivalent to the covariant representation induced by (ρ, V) . Since the representations of \mathcal{A} on the subspaces \mathcal{H}_j are pairwise non-quasi-equivalent by Lemma 4.11(ii) and the group G acts transitively on $\{\mathcal{H}_1, \dots, \mathcal{H}_n\}$, we obtain

$$H = \{g \in G \mid \rho \circ \alpha_g \approx \rho\}.$$

For (ii), note that $(\pi \rtimes U)(\mathcal{A} \rtimes G)'' \supseteq \pi(\mathcal{A})'' \supseteq \mathcal{Z}$, hence using the von Neumann Bicommutant Theorem, $(\pi \rtimes U)(\mathcal{A} \rtimes G)' = (\pi \rtimes U)(\mathcal{A} \rtimes G)''' \subseteq \mathcal{Z}' = \{p_1, \dots, p_n\}'$, and now the conclusion follows by Corollary 4.6. \square

Definition 4.13. For any C^* -algebra \mathcal{A} , we denote by $\text{Fact}(\mathcal{A})$ the class of all unitary equivalence classes $[\pi]$ of its factor representations π .

For the C^* -dynamical system (\mathcal{A}, G, α) , one has a natural action

$$G \times \text{Fact}(\mathcal{A}) \rightarrow \text{Fact}(\mathcal{A}), \quad (g, [\pi]) \mapsto [\pi \circ \alpha_{g^{-1}}].$$

We define the class $\mathcal{C}(\mathcal{A}, G, \alpha)$ of unitary equivalence classes $[(\pi_0, V)]$ of covariant representations (π_0, V) of C^* -dynamical systems of the form $(\mathcal{A}, G_0, \alpha|_{G_0})$, where $[\pi_0] \in \text{Fact}(\mathcal{A})$, $[\pi_0 \rtimes V] \in \text{Fact}(\mathcal{A} \rtimes G_0)$ and G_0 is the isotropy group in G of the quasi-equivalence class of π_0 . There is a natural action

$$G \times \mathcal{C}(\mathcal{A}, G, \alpha) \rightarrow \mathcal{C}(\mathcal{A}, G, \alpha), \quad (g, [(\pi_0, V)]) \mapsto [(\pi_0 \circ \alpha_{g^{-1}}, V \circ c_g^{-1}|_{gG_0g^{-1}})] \quad (5)$$

where $c_g: G \rightarrow G, h \mapsto ghg^{-1}$. This makes sense since gG_0g^{-1} is the isotropy group of the quasi-equivalence class of the factor representation $\pi_0 \circ \alpha_{g^{-1}}$ of \mathcal{A} , and the covariant representation $(\pi_0 \circ \alpha_{g^{-1}}, V \circ c_g^{-1}|_{gG_0g^{-1}})$ is a factor representation.

It is easily checked that two unitarily equivalent covariant representations give rise by induction to unitarily equivalent representations, and it then follows by Corollary 4.6 and Remark 4.7 that the induction of covariant representations defines a map

$$\Theta: \mathcal{C}(\mathcal{A}, G, \alpha) \rightarrow \text{Fact}(\mathcal{A} \rtimes_\alpha G). \quad (6)$$

Theorem 4.14. *For any C^* -dynamical system (\mathcal{A}, G, α) for which the group G is finite, the map Θ of (6) is constant on the orbits of the group action (5) and defines a bijective correspondence*

$$\overline{\Theta}: \mathcal{C}(\mathcal{A}, G, \alpha)/G \rightarrow \text{Fact}(\mathcal{A} \rtimes_{\alpha} G)$$

which preserves $$ -isomorphism classes of commutants, irreducibility, and the types I, II, III of factor representations.*

Proof. It follows by Lemma 4.12 that Θ is surjective.

Injectivity of $\overline{\Theta}$ follows by Lemma 4.5, where the subgroup G_0 is determined as the centralizer of a minimal central idempotent in $\pi(\mathcal{A})'$, and this determines the corresponding factor representation π_0 of \mathcal{A} . By Remark 4.7 the map Θ preserves $*$ -isomorphism classes of commutants, and we are done. \square

Remark 4.15. Resume the setting of Definition 4.13 and denote by $\mathcal{G}(\mathcal{A}, G, \alpha)$ the set of all subgroups $G_0 \subseteq G$ for which there exists $[\pi_0] \in \text{Fact}(\mathcal{A})$ such that G_0 is the isotropy group of the equivalence class of π_0 . If we consider the natural action of G on $\mathcal{G}(\mathcal{A}, G, \alpha)$ by conjugation, then one obtains a G -equivariant correspondence $\mathcal{C}(\mathcal{A}, G, \alpha) \rightarrow \mathcal{G}(\mathcal{A}, G, \alpha)$ that associates to every $[(\pi_0, V)]$ in $\mathcal{C}(\mathcal{A}, G, \alpha)$ the domain of definition of V . Using Theorem 4.14, this leads to a surjective correspondence

$$\text{Fact}(\mathcal{A} \rtimes_{\alpha} G) \simeq \mathcal{C}(\mathcal{A}, G, \alpha)/G \rightarrow \mathcal{G}(\mathcal{A}, G, \alpha)/G$$

which in particular defines an equivalence relation on $\text{Fact}(\mathcal{A} \rtimes_{\alpha} G)$ with finitely many equivalence classes.

This remark provides the following necessary criterion for unitary equivalence of two covariant factor representations (π_1, U_1) and (π_2, U_2) of (\mathcal{A}, G, α) : For $j = 1, 2$, assume that π_j is induced from some factor covariant representation (π_{j0}, V_j) of $(\mathcal{A}, G_j, \alpha|_{G_j})$ with $[\pi_{j0}] \in \text{Fact}(\mathcal{A})$ and G_j the isotropy group of the quasi-equivalence class of π_{j0} . If the subgroups G_1 and G_2 fail to be conjugated to each other in G , then the covariant factor representations (π_1, U_1) and (π_2, U_2) are not unitarily equivalent, and equivalently, $[\pi_1 \rtimes U_1] \neq [\pi_2 \rtimes U_2]$ in $\text{Fact}(\mathcal{A} \rtimes G)$.

Example 4.16. In the setting of Definition 4.13, consider the subclass $\mathcal{C}_0(\mathcal{A}, G, \alpha)$ that corresponds to $G_0 = \{1\}$. Hence the elements of $\mathcal{C}_0(\mathcal{A}, G, \alpha)$ can be identified with the unitary equivalence classes of factor $*$ -representations $\pi_0: \mathcal{A} \rightarrow B(\mathcal{H}_0)$ with the property that for every $1 \neq g \in G$ the factor representation $\pi_0 \circ \alpha_g$ is not quasi-equivalent to π_0 . Then the corresponding induced representation (π, U) of (\mathcal{A}, G, α) acts on the Hilbert space $\mathcal{H} = \ell^2(G, \mathcal{H}_0)$ consisting of all functions $f: G \rightarrow \mathcal{H}_0$ and is given by the formulas from Definition 4.1. The image of the equivalence class $[\pi_0]$ in $\mathcal{C}(\mathcal{A}, G, \alpha)/G$ is $\{\pi_0 \circ \alpha_g \mid g \in G\}$, and all the representations from this set induce the same element of $\text{Fact}(\mathcal{A} \rtimes_{\alpha} G)$ by Theorem 4.14.

In particular, this shows that there is no analogue of Theorem 4.14 with unitary equivalence classes of factor representations replaced by quasi-equivalence classes.

Example 4.17. To illustrate Definition 4.13 by a case which is opposite to Example 4.16, consider the subclass $\mathcal{C}_1(\mathcal{A}, G, \alpha)$ that corresponds to $G_0 = G$. Hence the elements of $\mathcal{C}_1(\mathcal{A}, G, \alpha)$ are the unitary equivalence classes of covariant factor representations (π_0, V) ,

where $\pi_0: \mathcal{A} \rightarrow B(\mathcal{H}_0)$ is also factor $*$ -representation. As (π_0, V) is a covariant representation of (\mathcal{A}, G, α) , it follows that, for every $g \in G$, the factor representation $\pi_0 \circ \alpha_g$ is quasi-equivalent (actually unitarily equivalent) to π_0 . Then the corresponding induced representation (π, U) of (\mathcal{A}, G, α) is given by the formulas from Definition 4.1 and is easily seen to be unitarily equivalent to (π_0, V) . The image of the equivalence class $[(\pi_0, V)]$ in $\mathcal{C}(\mathcal{A}, G, \alpha)/G$ is the singleton $\{[(\pi_0, V)]\}$, and this corresponds to $[\pi_0 \rtimes V] \in \text{Fact}(\mathcal{A} \rtimes_\alpha G)$ via Theorem 4.14.

For later use we also note a simple necessary criterion for unitary equivalence of factor representations $[\pi_{0j} \rtimes V_j]$, $j=1,2$, that belong to the image of $\mathcal{C}_1(\mathcal{A}, G, \alpha)$ in $\text{Fact}(\mathcal{A} \rtimes_\alpha G)$: if $[\pi_{01} \rtimes V_1] = [\pi_{02} \rtimes V_2]$, then $[\pi_{01}] = [\pi_{02}] \in \text{Fact}(\mathcal{A})$ and the unitary representations V_1 and V_2 of G are unitarily equivalent.

4.3 Classification of irreducible covariant representations

In the setting of Theorem 4.14, the definition of the class $\mathcal{C}(\mathcal{A}, G, \alpha)$ is somewhat involved, in the sense that for a covariant representation (π_0, V) of a C^* -dynamical system $(\mathcal{A}, G_0, \alpha|_{G_0})$, the unitary representation V of G_0 is rather implicitly described by the set of conditions $\pi_0 \in \text{Fact}(\mathcal{A})$, $\pi_0 \rtimes V \in \text{Fact}(\mathcal{A} \rtimes G_0)$, G_0 is the isotropy group of the quasi-equivalence class of π_0 . We will now discuss this issue for the subclass of irreducible covariant representation (π_0, V) , thus recovering the precise description of the dual space of $\mathcal{A} \rtimes G$ as in [Tak67], [AL10], and [Ka13a].

We will basically classify the factor representation ρ from Lemma 4.12 in the case when the covariant representation (π, U) is irreducible. See also [AL10, Th. 3.1] and the comments after [Ka13b, Th. 4.2]. If S is any group, \mathcal{H} is any Hilbert space, and $V: S \rightarrow B(\mathcal{H})$ and $\sigma: S \times S \rightarrow \mathbb{T}$ are maps satisfying the $V_{ts} = \sigma(t, s)V_t V_s$ for all $s, t \in S$, then we say that V is a σ -projective representation.

For the following proposition, we recall the existence of the minimal central projections from Lemmas 4.10 and 4.11.

Proposition 4.18. *Let (π, U) be an irreducible covariant representation of (\mathcal{A}, G, α) on \mathcal{H} . Let p_1, \dots, p_n be the minimal central projections in $\pi(\mathcal{A})'$. Then the following assertions hold:*

- (i) *For $j \in \{1, \dots, n\}$ let $d_j^2 := \dim(\pi(\mathcal{A})'p_j)$ and $G_j := \{g \in G \mid U_g p_j U_g^{-1} = p_j\}$. Then $\pi_j \cong \rho_j^{\oplus d_j}$ for an irreducible representation (ρ_j, \mathcal{K}_j) of \mathcal{A} and the covariant representation (π_j, V_j) of $(\mathcal{A}, G_j, \alpha|_{G_j})$ on \mathcal{H}_j is irreducible.*
- (ii) *$G_j = \{g \in G \mid \rho_j \circ \alpha_g \simeq \rho_j\}$ and $d_j = d_1$ for $j = 1, \dots, n$.*
- (iii) *The covariant representation (π_1, V_1) is unitarily equivalent to a covariant representation $(\rho_1 \otimes \mathbf{1}, V_1^{(1)} \otimes V_1^{(2)})$, where $V_1^{(1)}: G_1 \rightarrow \text{U}(\mathcal{K}_1)$ is a unitary σ -projective representation for a 2-cocycle $\sigma: G_1 \times G_1 \rightarrow \mathbb{T}$ whose cohomology class is uniquely determined, and $V_1^{(2)}: G_1 \rightarrow \text{U}_{d_1}(\mathbb{C})$ is a unitary irreducible $\bar{\sigma}$ -projective representation.*
- (iv) *The map Θ of (6) defines by restriction a bijection onto $(\mathcal{A} \rtimes G)^\wedge$ from the G -orbits of equivalence classes $[(\rho_1 \otimes \mathbf{1}, V_1^{(1)} \otimes V_1^{(2)})]$ as above, where $V_1^{(2)}$ can be any irreducible*

$\bar{\sigma}$ -projective unitary representation of the isotropy group G_1 of $[\rho_1] \in \hat{\mathcal{A}}$, and unitary inequivalent choices of $V_1^{(2)}$ lead to distinct equivalence classes $[(\rho_1 \otimes \mathbf{1}, V_1^{(1)} \otimes V_1^{(2)})]$.

Proof. (i) The covariant representation (π_j, V_j) of $(\mathcal{A}, G_j, \alpha|_{G_j})$ on \mathcal{H}_j is irreducible by Proposition 4.12(ii).

(ii) follows by Lemma 4.11(iii).

(iii) follows by [Tak67, Lemmas 5.1-5.2], where the C^* -algebra needs not be separable if the group is finite. In fact, in this case, [Tak67, Th. 2.6] (needed in the proof of [Tak67, Lemma 5.1]) is actually the trivial observation that, because

$$G_1 = \{g \in G \mid \rho_1 \circ \alpha_g \simeq \rho_1\},$$

there exist a 2-cocycle $\sigma: G_1 \times G_1 \rightarrow \mathbb{T}$, and a unitary σ -projective representation $V_1^{(1)}: G_1 \rightarrow \mathcal{U}(\mathcal{K}_1)$, with

$$(\rho_1 \circ \alpha_g)(x) = V_1^{(1)}(g)\rho_1(x)V_1^{(1)}(g)^* \quad \text{for } g \in G_1, x \in \mathcal{A}.$$

The cohomology class $[\sigma]$ is uniquely determined because the corresponding central \mathbb{T} -extension of G_1 is isomorphic to the pullback of the \mathbb{T} -extension of $\text{PU}(\mathcal{K}_1) := \mathcal{U}(\mathcal{K}_1)/\mathbb{T}\mathbf{1}$ under the corresponding projective representation. Then one has

$$(V_1^{(1)}(g) \otimes \mathbf{1})\pi_1(x)(V_1^{(1)}(g) \otimes \mathbf{1})^* = (\pi_1 \circ \alpha_g)(x) = V_g\pi_1(x)V_g,$$

hence $(V_1^{(1)}(g) \otimes \mathbf{1})^*V_g$ belongs to the commutant of $\pi_1 \cong \rho_1^{\oplus d_1}$, hence there exists a $V_1^{(2)} \in \mathcal{U}_{d_1}(\mathbb{C})$ with $(V_1^{(1)}(g) \otimes \mathbf{1})^*V_g = \mathbf{1} \otimes (V_1^{(2)}(g))$. Since V is a unitary representation, $V \simeq V_1^{(1)} \otimes V_1^{(2)}$, and $V_1^{(1)}$ is a σ -projective representation, it follows that $V_1^{(2)}$ is a $\bar{\sigma}$ -projective unitary representation of G_1 , which is also irreducible, exactly as in the proof of [Tak67, Lemma 5.2].

Conversely, given $[\rho_1] \in \hat{\mathcal{A}}$, one determines its isotropy group G_1 with respect to the action of G on $\hat{\mathcal{A}}$, then a cocycle $\sigma \in Z^2(G_1, \mathbb{T})$ and a σ -projective representation $V_1^{(1)}: G_1 \rightarrow \mathcal{U}(\mathcal{K}_1)$, and then every irreducible $\bar{\sigma}$ -projective unitary representation $V_1^{(2)}$ gives rise to a covariant representation $(\rho_1 \otimes \mathbf{1}, V_1^{(1)} \otimes V_1^{(2)})$ of $(\mathcal{A}, G_1, \alpha|_{G_1})$ which is an irreducible representation and induces an irreducible covariant representation of (\mathcal{A}, G, α) . Thus (iv) follows by Theorem 4.14, and we are done. \square

4.4 Representations of fixed-point subalgebras

We will need the following slight generalization of the well-known fact that factor $*$ -representations are nondegenerate.

Lemma 4.19. *Let \mathcal{B}_0 be a $*$ -subalgebra of some $*$ -algebra \mathcal{B} . Then, for any factor representation $\pi: \mathcal{B} \rightarrow B(\mathcal{H})$, one has $\mathcal{B}_0 \not\subset \ker \pi$ if and only if the subspace $\text{span } \pi(\mathcal{B}\mathcal{B}_0)\mathcal{H}$ is dense in \mathcal{H} .*

Proof. The hypothesis that π is a factor representation is equivalent to the fact that there exists no nontrivial closed linear subspace of \mathcal{H} that it is invariant both under $\pi(\mathcal{B})$ and its

commutant $\pi(\mathcal{B})'$, since the orthogonal projection on such a subspace would belong to the center $\pi(\mathcal{B})' \cap \pi(\mathcal{B})''$ of $\pi(\mathcal{B})''$.

We may assume $\mathcal{H} \neq \{0\}$ and the assertion will follow by the above observation as soon as we will have proved that if $\mathcal{B}_0 \not\subseteq \ker \pi$ then $\pi(\mathcal{B}\mathcal{B}_0)\mathcal{H} \neq \{0\}$, since this subset of \mathcal{H} is invariant under $\pi(\mathcal{B})$ and $\pi(\mathcal{B})'$. In fact, since $\mathcal{B}_0 \not\subseteq \ker \pi$, there exists $b_0 \in \mathcal{B}_0$ with $\pi(b_0) \neq 0$ and then $\pi(b_0^*)\pi(b_0)\mathcal{H} \subseteq \pi(\mathcal{B})\pi(\mathcal{B}_0)\mathcal{H}$ is non-zero. \square

Proposition 4.20. *Let $\mathcal{B}_0 \subseteq \mathcal{B}$ be a $*$ -subalgebra of some Banach $*$ -algebra with approximate unit, with $\mathcal{B}_0\mathcal{B}\mathcal{B}_0 \subseteq \mathcal{B}_0$. Denote by $\text{Fact}_{\mathcal{B}_0}(\mathcal{B})$ the class of all $[\pi] \in \text{Fact}(\mathcal{B})$ with $\mathcal{B}_0 \not\subseteq \ker \pi$ and denote by $\Psi([\pi])$ the unitary equivalence class of the \mathcal{B}_0 -nondegenerate part of $\pi|_{\mathcal{B}}$ for $[\pi] \in \text{Fact}_{\mathcal{B}_0}(\mathcal{B})$. Then the correspondence*

$$\Psi: \text{Fact}_{\mathcal{B}_0}(\mathcal{B}) \rightarrow \text{Fact}(\mathcal{B}_0)$$

is a bijection and a homeomorphism with respect to the regional topologies, which preserves $$ -isomorphism classes of commutants, irreducibility, and the types I, II, III of factor representations.*

Proof. This follows by [FeDo88, vol. II, XI.7.6, Prop. and Rem. 1], using the above Lemma 4.19. \square

For the following statement we recall that for any C^* -algebra \mathcal{B} we denote by $M(\mathcal{B})$ its multiplier algebra, which is a unital C^* -algebra with a canonical embedding $\mathcal{B} \hookrightarrow M(\mathcal{B})$, and one has $\mathcal{B} = M(\mathcal{B})$ if and only if \mathcal{B} is unital.

Proposition 4.21. *Denote $\mathcal{A}^G := \{x \in \mathcal{A} \mid (\forall g \in G) \alpha_g(x) = x\}$ and define $\tilde{\mathcal{A}} := \mathcal{A} + \mathbb{C}\mathbf{1} \subseteq M(\mathcal{A})$. Let $p \in \ell^1(G, \tilde{\mathcal{A}}; \alpha) \subseteq M(\mathcal{A} \rtimes_{\alpha} G)$ be the constant function that is equal to $\mathbf{1} \in \tilde{\mathcal{A}}$ at every point of G . Then the following assertions hold:*

- (i) *One has $p = p^* = p^2$, and by restriction and co-restriction the injective linear map $\mathcal{A} \rightarrow \mathcal{A} \rtimes_{\alpha} G$, $x \mapsto \alpha(\cdot, x)$, yields a $*$ -isomorphism $\iota: \mathcal{A}^G \rightarrow p(\mathcal{A} \rtimes_{\alpha} G)p$.*
- (ii) *For every factor covariant representation (π, U) of (\mathcal{A}, G, α) , we will write $(\pi \rtimes U)^{\sim}$ for the canonical extension of $\pi \rtimes U$ from $\mathcal{A} \rtimes_{\alpha} G$ to $M(\mathcal{A} \rtimes_{\alpha} G)$, and by $(\pi \rtimes U)^G$ the representation of \mathcal{A}^G obtained as the nondegenerate part of $(\pi \rtimes U)^{\sim} \circ \iota$. We also define the following open subset of $\text{Fact}(\mathcal{A} \rtimes_{\alpha} G)$,*

$$\text{Fact}_p(\mathcal{A} \rtimes_{\alpha} G) := \{[\pi \rtimes U] \in \text{Fact}(\mathcal{A} \rtimes_{\alpha} G) \mid (\pi \rtimes U)^{\sim}(p) \neq 0\}.$$

Then the map

$$\text{Fact}_p(\mathcal{A} \rtimes_{\alpha} G) \rightarrow \text{Fact}(\mathcal{A}^G), \quad [\pi \rtimes U] \mapsto [(\pi \rtimes U)^G]$$

is a well-defined homeomorphism which preserves $$ -isomorphism classes of commutants.*

- (iii) *One has $[\pi \rtimes U] \in \text{Fact}_p(\mathcal{A} \rtimes_{\alpha} G)$ if and only if*

$$[\pi \rtimes U] \in \text{Fact}(\mathcal{A} \rtimes_{\alpha} G) \quad \text{and} \quad \sum_{s \in G} U_s \neq 0,$$

and in this case $P_U := \frac{1}{|G|} \sum_{s \in G} U_s$ is the orthogonal projection onto the essential space of the $$ -representation $(\pi \rtimes U) \circ \iota$ of \mathcal{A}^G .*

(iv) Let $G_0 \subseteq G$ be any subgroup and (π_0, V) be any covariant representation of the C^* -dynamical system $(\mathcal{A}, G_0, \alpha|_{G_0})$ for which the corresponding induced covariant representation (π, U) of (G, \mathcal{A}, α) is irreducible. Then $[\pi \rtimes U] \in \text{Fact}_p(\mathcal{A} \rtimes_\alpha G)$ if and only if $\sum_{s \in G_0} V_s \neq 0$. In particular, this is always the case if $G_0 = \{1\}$.

(v) The C^* -algebra \mathcal{A} is of type I if and only if \mathcal{A}^G is, and if and only if $\mathcal{A} \rtimes_\alpha G$ is.

Proof. Since G is in particular a compact group, Assertion (i) follows by [Ro79] (see also [Bl06, II.10.4.18]).

For Assertion (ii), denote $\mathcal{B} := \mathcal{A} \rtimes_\alpha G = \ell^1(G, \mathcal{A}; \alpha)$, $\mathcal{B}_0 := p\mathcal{B}p$, and note that, since $p = p^* = p^2 \in M(\mathcal{B})$, $p\mathcal{B}p$ is a hereditary subalgebra of \mathcal{B} , that is, $\mathcal{B}_0\mathcal{B}\mathcal{B}_0 \subseteq \mathcal{B}_0$. Then the assertion follows by Proposition 4.20 along with Assertion (i).

For Assertion (iii), recall that if (π, U) is a covariant representation of (\mathcal{A}, G, α) , then the corresponding representation of $\mathcal{B} = \ell^1(G, \mathcal{A}; \alpha)$ is

$$\pi \rtimes U: \mathcal{B} \rightarrow B(\mathcal{H}), \quad (\pi \rtimes U)(f) = \frac{1}{|G|} \sum_{s \in G} \pi(f(s))U_s.$$

If (π, U) is a factor representation, then also $\pi \rtimes U: \mathcal{B} \rightarrow B(\mathcal{H})$ is a factor representation hence it is nondegenerate, and then it extends to $M(\mathcal{B}) \supseteq \ell^1(G, \tilde{\mathcal{A}}; \alpha)$. By the above formula one then obtains for the constant function $p: G \rightarrow \tilde{\mathcal{A}}$,

$$(\pi \rtimes U)(p) = \frac{1}{|G|} \sum_{s \in G} U_s.$$

Therefore the condition $(\pi \rtimes U)(p) \neq 0$ is equivalent to $\sum_{s \in G} U_s \neq 0$.

For Assertion (iv), let $\{1 = g_0, g_1, \dots, g_n\}$ be a complete system of representatives of the G_0 -left cosets sets in G . Note that

$$\left(\sum_{g \in G} U_g \right) \mathcal{H} = \{v \in \mathcal{H} \mid (\forall g \in G) U_g v = v\} =: \mathcal{H}^G$$

and similarly

$$\left(\sum_{s \in G_0} V_s \right) \mathcal{H}_0 = \{v_0 \in \mathcal{H}_0 \mid (\forall s \in G_0) V_s v_0 = v_0\} =: \mathcal{H}_0^{G_0},$$

hence we must show that $\mathcal{H}^G \neq \{0\}$ if and only if $\mathcal{H}_0^{G_0} \neq \{0\}$, which follows by Remark 4.3.

Finally, Assertion (iv) follows by [Ri80, Th. 4.1], and we are done. \square

Our next aim is to describe the dual space $\widehat{\mathcal{A}^G}$ using Proposition 4.21. In order to simplify the corresponding statement (Theorem 4.23) we make the following definition.

Definition 4.22. For the C^* -dynamical system (\mathcal{A}, G, α) , let $\pi_0: \mathcal{A} \rightarrow B(\mathcal{H}_{\pi_0})$ be an arbitrary irreducible $*$ -representation and

$$G_{\pi_0} := \{g \in G \mid \pi_0 \circ \alpha_g \simeq \pi_0\}.$$

Let $\omega_0: G_{\pi_0} \times G_{\pi_0} \rightarrow \mathbb{T}$ be a 2-cocycle and $V_0: G_{\pi_0} \rightarrow U(\mathcal{H}_{\pi_0})$ be any ω_0 -projective unitary representation satisfying

$$(\forall g \in G_{\pi_0})(\forall a \in \mathcal{A}) \quad V_0(g)\pi_0(a)V_0(g)^* = \pi_0(\alpha_g(a)).$$

Let $\Gamma(\mathcal{A}, G, \alpha)$ be the class of all covariant representations (π, V) of $(\mathcal{A}, G_{\pi_0}, \alpha|_{G_{\pi_0}})$ obtained as

$$\pi := \pi_0 \otimes \text{id}_{\mathcal{H}_{\overline{U_0}}} \quad \text{and} \quad V := V_0 \otimes \overline{U_0}$$

where $U_0: G_{\pi_0} \rightarrow B(\mathcal{H}_{U_0})$ is an arbitrary irreducible ω_0 -projective unitary representation, and let $\Gamma_0(\mathcal{A}, G, \alpha) \subseteq \Gamma(\mathcal{A}, G, \alpha)$ be the subclass defined by imposing the additional condition

$$\sum_{g \in G_{\pi_0}} V_0(g) \otimes \overline{U_0}(g) \neq 0. \quad (7)$$

Then the set $\tilde{\Gamma}_0(\mathcal{A}, G, \alpha)$ of all unitary equivalence classes in $\Gamma_0(\mathcal{A}, G, \alpha)$, is naturally acted on by G via α and via the conjugation action of G on itself as in (5) (see also [Tak67, Th. 7.2], [Ka13a, Sect. 2]), and we denote by $\tilde{\Gamma}_0/G$ the corresponding quotient set.

Finally, we define

$$\mathcal{I}: \tilde{\Gamma}_0(\mathcal{A}, G, \alpha)/G \rightarrow \widehat{\mathcal{A}^G} \quad (8)$$

to be the correspondence that takes the unitary equivalence class $[(\pi, V)]$ of a covariant representation of $(\mathcal{A}, G_{\pi_0}, \alpha|_{G_{\pi_0}})$ to the unitary equivalence class of $\text{Ind}_{G_{\pi_0}}^G V$ and then composing that correspondence with the map $[\Pi] \mapsto [\Pi^G]$ given by Proposition 4.21(ii).

The following result should be compared with [Ka13a, Th. 3.3] (or [Tak67, Th. 7.2] if the C^* -algebra \mathcal{A} is of type I).

Theorem 4.23. *For every C^* -dynamical system (\mathcal{A}, G, α) , whose group G is finite, the map $\mathcal{I}: \tilde{\Gamma}_0(\mathcal{A}, G, \alpha)/G \rightarrow \widehat{\mathcal{A}^G}$ is a bijection.*

Proof. Using the notation of Definition 4.22, one can see that Proposition 4.18 establishes the bijection (8) from the set of G -orbits in $\tilde{\Gamma}(\mathcal{A}, G, \alpha)$ onto $(\mathcal{A} \rtimes G)^\wedge$.

In some more detail, to check surjectivity of \mathcal{I} , we use our Lemma 4.12 to show that every irreducible covariant representation of (\mathcal{A}, G, α) is induced from a factor representation of type I. Now the conclusion follows by Proposition 4.21(ii) and (iv). \square

5 Representations of symmetric tensor powers

It follows by [BN15] that the representations of exponential C^* -algebras $e^{\mathcal{A}} = \bigoplus_{n \in \mathbb{N}_0}^{c_0} S^n(\mathcal{A})$ play an important role in the representation theory of ball semigroups. Since any exponential C^* -algebra is the c^0 -direct sum of the symmetric tensor powers $S^n(\mathcal{A}) = (\mathcal{A}^{\otimes n})^{S_n}$ of a C^* -algebra, one needs to understand the representations of the algebras $S^n(\mathcal{A})$ in terms of the representations of \mathcal{A} .

Definition 5.1. If \mathcal{A} is a C^* -algebra, then for every permutation $\sigma \in S_n$ we denote by $\alpha_\sigma: \mathcal{A}^{\otimes n} \rightarrow \mathcal{A}^{\otimes n}$ its corresponding permutation action, and we denote by $(\mathcal{A}^{\otimes n}, S_n, \alpha)$ the C^* -dynamical system obtained in this way.

We record the following fact for later use.

Lemma 5.2. *If $n \geq 2$, $\mathcal{A}_1, \dots, \mathcal{A}_n$ are C^* -algebras and $\mathcal{A} := \mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_n$, then the map*

$$\widehat{\mathcal{A}_1} \times \dots \times \widehat{\mathcal{A}_n} \rightarrow \widehat{\mathcal{A}}, \quad ([\pi_1], \dots, [\pi_n]) \mapsto [\pi_1 \otimes \dots \otimes \pi_n] \quad (9)$$

is injective, and it is also surjective if at least $n - 1$ of the C^ -algebras $\mathcal{A}_1, \dots, \mathcal{A}_n$ are of type I.*

Proof. If $n = 2$, the assertion follows by [OT66, p. 333], or [Bl06, IV.3.4.22/26], then use induction. \square

5.1 The case of C^* -algebras of type I

We are now in a position to describe the representations of the algebras $S^n(\mathcal{A})$ in terms of the representations of \mathcal{A} for any C^* -algebra \mathcal{A} of type I, by basically specializing Theorem 4.23 to the C^* -dynamical system $(\mathcal{A}^{\otimes n}, S_n, \alpha)$. For the following statement we recall that a *full corner* of a C^* -algebra \mathcal{A} is any subalgebra of the form $p\mathcal{A}p$, where $p = p^2 = p^* \in M(\mathcal{A})$, that is not contained in any closed two-sided ideal of \mathcal{A} , (see [Br77]).

Theorem 5.3. *Let \mathcal{A} be a C^* -algebra of type I and $n \geq 1$. For the C^* -dynamical system $(\mathcal{A}^{\otimes n}, S_n, \alpha)$, define the projection $p \in M(\mathcal{A}^{\otimes n} \rtimes_{\alpha} S_n)$ and the $*$ -isomorphism*

$$\iota: S^n(\mathcal{A}) \rightarrow p(\mathcal{A}^{\otimes n} \rtimes_{\alpha} S_n)p$$

as in Proposition 4.21(i). Then the following assertions hold:

- (i) *The hereditary subalgebra $p(\mathcal{A}^{\otimes n} \rtimes_{\alpha} S_n)p$ is a full corner of $\mathcal{A}^{\otimes n} \rtimes_{\alpha} S_n$.*
- (ii) *For every irreducible representation Π of $\mathcal{A}^{\otimes n} \rtimes_{\alpha} S_n$ one has $\tilde{\Pi}(p) \neq 0$.*
- (iii) *The map $[\Pi] \mapsto [\Pi^G]$ from Proposition 4.21(ii) is a homeomorphism from $(\mathcal{A}^{\otimes n} \rtimes_{\alpha} S_n)^{\wedge}$ onto $S^n(\mathcal{A})^{\wedge}$.*
- (iv) *For a partition $q_1 + \dots + q_m = n$, let $\mathbf{q} := (q_1, \dots, q_m)$ and $S_{\mathbf{q}} := S_{q_1} \times \dots \times S_{q_m} \subseteq S_n$. Let Γ_0 be the family of covariant representations (Π_1, W_1) of $(\mathcal{A}^{\otimes n}, S_{\mathbf{q}}, \alpha|_{S_{\mathbf{q}}})$ obtained as*

$$\Pi_1 := \pi_1^{\otimes q_1} \otimes \dots \otimes \pi_m^{\otimes q_m} \otimes \text{id}_{\mathcal{H}_{U_{01}}} \otimes \dots \otimes \text{id}_{\mathcal{H}_{U_{0m}}} \quad \text{and} \quad W_1 := V_0 \otimes \overline{U_{01}} \otimes \dots \otimes \overline{U_{0m}}$$

where $\pi_k: \mathcal{A} \rightarrow B(\mathcal{H}_{\pi_k})$ for $k = 1, \dots, m$ are mutually inequivalent irreducible $$ -representations, $V_0: S_{\mathbf{q}} \rightarrow U(\mathcal{H}_{\pi_1}^{\otimes q_1} \otimes \dots \otimes \mathcal{H}_{\pi_m}^{\otimes q_m})$ is the (external) tensor product of permutation representations, and $U_{0k}: S_{q_k} \rightarrow U(\mathcal{H}_{U_{0k}})$ is an arbitrary irreducible unitary representation for $k = 1, \dots, m$. Then the set $\tilde{\Gamma}_0$ of all unitary equivalence classes in Γ_0 , is naturally acted upon by S_n via α , and the bijection $\tilde{\Gamma}_0/S_n \rightarrow (\mathcal{A}^{\otimes n} \rtimes_{\alpha} S_n)^{\wedge}$ takes every covariant representation (Π_1, W_1) to its induced covariant representation of $(\mathcal{A}^{\otimes n}, S_n, \alpha)$.*

Proof. (i) By [Br77, Lemma 1] this is a consequence of (ii).

(ii) Since \mathcal{A} is a C^* -algebra of type I, it follows by Lemma 5.2 that the map (9) with $\mathcal{A}_1 = \cdots = \mathcal{A}_n = \mathcal{A}$ is bijective for every $n \geq 1$. This implies that, for an irreducible representation $\Pi_0: \mathcal{A}^{\otimes n} \rightarrow B(\mathcal{H}_{\Pi_0})$, there exist uniquely determined irreducible representations π_1, \dots, π_n of \mathcal{A} with $\Pi_0 \cong \pi_1 \otimes \cdots \otimes \pi_n$. After conjugating with a suitable permutation, we may assume

$$\Pi_0 \cong \rho_1^{\otimes q_1} \otimes \cdots \otimes \rho_m^{\otimes q_m},$$

so that the isotropy group of $[\Pi_0]$ is conjugate to $S_{\mathbf{q}}$.

Then, in the notation of Theorem 4.23, one has $\omega_0 \equiv 1$ and $V_0: (S_n)_{\Pi_0} \rightarrow U(\mathcal{H}_{\Pi_0})$ is the (external) tensor product of the permutation representations $V_{0k}: S_{q_k} \rightarrow U(\mathcal{H}_{\rho_k^{\otimes q_k}})$. Since $\omega_0 \equiv 1$, it follows that $U_0: (S_n)_{\Pi_0} \rightarrow U(\mathcal{H}_{U_0})$ is an arbitrary irreducible unitary representation satisfying (7). By [OT66, Cor. to Th. 2], there exist uniquely determined irreducible unitary representations $U_{0k}: S_{q_k} \rightarrow U(\mathcal{H}_{U_{0k}})$ for $k = 1, \dots, m$ whose (external) tensor product is U_0 . Then, using the fact that, if $T_k \in B(\mathcal{H}_{0k})$ for $k = 1, \dots, m$, then $T_1 \otimes \cdots \otimes T_m \neq 0$ if and only if $T_k \neq 0$ for $k = 1, \dots, m$, it is easily seen that condition (7) is equivalent to

$$T_k := \sum_{\sigma \in S_{q_k}} V_{0k}(\sigma) \otimes \overline{U_{0k}}(\sigma) \neq 0 \text{ for } k = 1, \dots, m.$$

To see that the above condition is satisfied, note that the irreducibility of the representation U_{0k} implies

$$\frac{1}{q_k!} \sum_{\sigma \in S_{q_k}} U_{0k}(\sigma) = \mathbf{1} \text{ for } k = 1, \dots, m.$$

Hence for any nonzero vector $v \in \mathcal{H}_{U_{0k}}$ and for every nonzero symmetric tensor $w \in S^{q_k}(\mathcal{H}_{\pi_{j_k^0}}) \subseteq \mathcal{H}_{\pi_{j_k^0}}^{\otimes q_k}$ one has

$$\begin{aligned} 0 \neq w \otimes (q_k!v) &= w \otimes \sum_{\sigma \in S_{q_k}} \overline{U_{0k}}(\sigma)v = \sum_{\sigma \in S_{q_k}} V_{0k}(\sigma)w \otimes \overline{U_{0k}}(\sigma)v \\ &= \left(\sum_{\sigma \in S_{q_k}} V_{0k}(\sigma) \otimes \overline{U_{0k}}(\sigma) \right) (w \otimes v). \end{aligned}$$

This completes the proof of Assertion (ii).

(iii) follows by Theorem 4.23, and we are done. \square

Example 5.4. Consider the C^* -algebra $\mathcal{A} = \mathfrak{S}_{\infty}(\mathcal{H})$ of compact operators on some infinite dimensional complex Hilbert space. Then it is known that $\widehat{\mathcal{A}} = \{[\pi]\}$, where π is the tautological representation of \mathcal{A} . Then, for any $n \geq 2$, in Theorem 5.3 one has $m = 1$, and it follows that $S^n(\mathcal{A})^\wedge$ is parametrized by $\widehat{S_n}$, just as the holomorphic Schur–Weyl representations. See Subsection 5.2 below for more details.

5.2 Links to Schur–Weyl theory for irreducible representations

Let \mathcal{A} be any C^* -algebra and recall from the introduction that its *unitary group* is

$$U(\mathcal{A}) := (\mathbf{1} + \mathcal{A}) \cap U(M(\mathcal{A})),$$

where $M(\mathcal{A})$ is the multiplier C^* -algebra of \mathcal{A} , and $M(\mathcal{A}) = \mathcal{A}$ if \mathcal{A} is unital. It follows from [BN12] that to every irreducible $*$ -representation $\pi: \mathcal{A} \rightarrow B(\ell^2(J))$ there corresponds an infinite family of unitary irreducible representations $\pi_\lambda^{\mathcal{A}}$ of $U(\mathcal{A})$ obtained by Schur–Weyl theory (see [BN12, Def. 4.1]), where λ belongs to the additive group $\mathcal{P}_J \cong \mathbb{Z}^{(J)}$ of all finitely supported \mathbb{Z} -valued functions on J . Here a set $J = J_{[\pi]}$ is fixed for every equivalence class $[\pi] \in \widehat{\mathcal{A}}$. We then define the *Schur–Weyl map* of \mathcal{A} as

$$\Psi_{\mathcal{A}}: \bigsqcup_{[\pi] \in \widehat{\mathcal{A}}} \mathcal{P}_{J_{[\pi]}}/S_{(J_{[\pi]})} \rightarrow U(\mathcal{A})^\wedge, \quad [\lambda] \mapsto [\pi_\lambda^{\mathcal{A}}] \quad \text{for } [\lambda] \in \mathcal{P}_{J_{[\pi]}}/S_{(J_{[\pi]})} \text{ and } [\pi] \in \widehat{\mathcal{A}}, \quad (10)$$

where the notation is as in [BN12, Th. 3.2]. In particular, for $\lambda \in \mathcal{P}_J$, we denote by $[\lambda]$ its equivalence class modulo the natural action of the group $S_{(J)}$ of finitely supported permutations of J . As an application of Theorem 4.14 (via Theorem 4.23) we will show in Theorem 5.8 that *the Schur–Weyl map $\Psi_{\mathcal{A}}$ is always injective* but, if \mathcal{A} is separable and non-isomorphic to $\mathcal{K}(\ell^2(\mathbb{N}))$, then $\widehat{\mathcal{A}}$ contains at least two elements and this implies that $\Psi_{\mathcal{A}}$ is not surjective.

On the other hand, using [BN15, Th. 6.8], one easily sees that if $\lambda_j \geq 0$ for every j and $n = \sum_j \lambda_j$, then the representation $\pi_\lambda^{\mathcal{A}}$ uniquely extends to an irreducible $*$ -representation of $S^n(\mathcal{A})$, denoted again by $\pi_\lambda^{\mathcal{A}}$ and termed a *Schur–Weyl representation* in the following, where by extension we actually mean a factorization through the $*$ -morphism

$$U(\mathcal{A}) \hookrightarrow S^n(\mathcal{A}), \quad a \mapsto a^{\otimes n}.$$

To begin with, we will show in Proposition 5.7 how the equivalence classes $[\pi_\lambda^{\mathcal{A}}] \in S^n(\mathcal{A})^\wedge$ can be recovered in the setting of the above Theorem 4.14. To this end we need the following general lemma.

Lemma 5.5. *Let (π, U) be a covariant representation of a C^* -dynamical system (\mathcal{B}, G, α) , with the finite group G , on a Hilbert space \mathcal{H} for which π is irreducible. For any unitary irreducible representation $\lambda: G \rightarrow B(\mathcal{V})$ define the Hilbert space $\mathcal{H}^\lambda := B_2(\mathcal{V}, \mathcal{H}) \simeq \mathcal{H} \otimes \mathcal{V}^*$ and the $*$ -representation $\pi^\lambda: \mathcal{B} \rightarrow B(\mathcal{H}^\lambda)$, $\pi^\lambda(x)T := \pi(x)T$. Then the following assertions hold:*

- (i) *The pair $(\pi^\lambda, U \otimes \bar{\lambda})$ is an irreducible covariant representation of (\mathcal{B}, G, α) .*
- (ii) *Let $\iota: \mathcal{B}^G \rightarrow \mathcal{B} \subseteq \mathcal{B} \rtimes G$ be the canonical embedding as in Proposition 4.21(i). If $\text{Hom}_G(\mathcal{V}, \mathcal{H}) \neq \{0\}$, then this is the essential space of $(\pi^\lambda \rtimes (U \otimes \bar{\lambda})) \circ \iota: \mathcal{B}^G \rightarrow B(\mathcal{H}^\lambda)$ and the corresponding nondegenerate representation of \mathcal{B}^G on $\text{Hom}_G(\mathcal{V}, \mathcal{H})$ is irreducible and a subrepresentation of $\pi^\lambda|_{\mathcal{B}^G}$.*

Proof. For every $g \in G$ and $x \in \mathcal{B}$ one has

$$(U \otimes \bar{\lambda})(g)\pi^\lambda(x)(U \otimes \bar{\lambda})(g^{-1}) = (U_g\pi(x)U_{g^{-1}}) \otimes \text{id}_{\mathcal{V}^*} = \pi(\alpha_g(x)) \otimes \text{id}_{\mathcal{V}^*} = \pi^\lambda(\alpha_g(x)),$$

so that $(\pi^\lambda, U \otimes \bar{\lambda})$ is a covariant representation of (\mathcal{B}, G, α) .

To prove that $(\pi^\lambda, U \otimes \bar{\lambda})$ is irreducible, we must check that $\pi^\lambda(\mathcal{B})' \cap (U \otimes \bar{\lambda})'_G = \mathbb{C} \text{id}_{\mathcal{H} \otimes \mathcal{V}^*}$. One has $\pi^\lambda(\mathcal{B})' = (\pi(\mathcal{B}) \otimes \mathbb{C} \text{id}_{\mathcal{V}^*})' = \pi(\mathcal{B})' \otimes B(\mathcal{V}^*) = \mathbb{C} \text{id}_{\mathcal{H}} \otimes B(\mathcal{V}^*)$ because π is irreducible. Hence we must prove that if $T \in B(\mathcal{V}^*)$ satisfies $\text{id}_{\mathcal{H}} \otimes T \in (U \otimes \bar{\lambda})'_G$, then $T \in \mathbb{C} \text{id}_{\mathcal{V}^*}$. The

condition on T is equivalent to $T \in \overline{\lambda}_G$ hence, using the hypothesis that λ is an irreducible representation hence so is $\overline{\lambda}$, one obtains $T \in \mathbb{C} \text{id}_{\mathcal{V}^*}$, and this completes the proof of the first assertion.

For the second assertion, it follows by Proposition 4.21(ii) and (iii) that $\text{Hom}_G(\mathcal{V}, \mathcal{H})$, that is, $(\mathcal{H} \otimes \mathcal{V}^*)^G$, is the essential space of $(\pi^\mathcal{V} \rtimes (U \otimes \overline{\lambda})) \circ \iota$, and the corresponding nondegenerate representation of \mathcal{B}^G is irreducible because $\pi^\mathcal{V} \rtimes (U \otimes \overline{\lambda})$ is irreducible. To see that this irreducible representation of \mathcal{B}^G is a subrepresentation of $\pi^\mathcal{V}|_{\mathcal{B}^G}$, that is, it is equal to $x \mapsto \pi^\mathcal{V}(x)|_{\text{Hom}_G(\mathcal{V}, \mathcal{H})}$, note that, for $x \in \mathcal{B}^G$, one has

$$\begin{aligned} ((\pi^\mathcal{V} \rtimes (U \otimes \overline{\lambda})) \circ \iota)(x) &= \frac{1}{|G|} \sum_{g \in G} (\pi^\mathcal{V}(\iota(x)(g)))(U \otimes \overline{\lambda})_g = \frac{1}{|G|} \sum_{g \in G} \pi^\mathcal{V}(x)(U \otimes \overline{\lambda})_g \\ &= \pi^\mathcal{V}(x)P = P\pi^\mathcal{V}(x), \end{aligned}$$

where $P := \frac{1}{|G|} \sum_{g \in G} (U \otimes \overline{\lambda})_g$ is the orthogonal projection of $\mathcal{H} \otimes \mathcal{V}^*$ onto $(\mathcal{H} \otimes \mathcal{V}^*)^G$. This completes the proof. \square

Remark 5.6. In connection with Lemma 5.5, we note that the factor covariant representations (π, U) of (\mathcal{B}, G, α) for which the irreducible decomposition of U contains a fixed irreducible representation λ of G were classified in [La80, Th. 1] even for compact groups G .

Proposition 5.7. *Let \mathcal{A} be a unital C^* -algebra and fix some integer $n \geq 1$ and a unitary irreducible representation $\lambda: S_n \rightarrow B(\mathcal{V})$. For any irreducible $*$ -representation $\pi: \mathcal{A} \rightarrow B(\mathcal{H})$ denote by $W(\pi): S_n \rightarrow B(\mathcal{H}^{\otimes n})$ the corresponding permutation representation and define*

$$(\pi^{\otimes n})^\mathcal{V}: \mathcal{A}^{\otimes n} \rightarrow B(\mathcal{H}^{\otimes n} \otimes \mathcal{V}^*), \quad (\pi^{\otimes n})^\mathcal{V}(x) := \pi^{\otimes n}(x) \otimes \text{id}_{\mathcal{V}^*}.$$

Then $((\pi^{\otimes n})^\mathcal{V}, W(\pi) \otimes \overline{\lambda})$ is an irreducible covariant representation of $(\mathcal{A}^{\otimes n}, S_n, \alpha)$ and the image of its S_n -orbit through the map \mathcal{I} of Theorem 4.23 is $[\pi_\lambda^{\mathcal{A}}] \in S^n(\mathcal{A})^\wedge \hookrightarrow U(\mathcal{A})^\wedge$.

Proof. The covariant representation $((\pi^{\otimes n})^\mathcal{V}, W(\pi) \otimes \overline{\lambda})$ is irreducible by Lemma 5.5, because $\pi^{\otimes n}: \mathcal{A}^{\otimes n} \rightarrow B(\mathcal{H}^{\otimes n})$ is irreducible. Using the notation of Definition 4.22, one has $\pi_0 = \pi^{\otimes n}$ and $G = G_{\pi_0} = S_n$, hence $\text{Ind}_{G_{\pi_0}}^G(\pi_0) = \pi_0$. Moreover, $\omega_0 \equiv 1$, $V_0 = W(\pi)$, and $U_0 = \lambda$. Condition (7) is satisfied, since this can be proved just as the relation $T_k \neq 0$ in the last part of the proof of Theorem 5.3(ii). The image of the S_n -orbit of the equivalence class of $(\pi^\mathcal{V}, W(\pi) \otimes \overline{\lambda})$ under \mathcal{I} is the equivalence class of the nondegenerate part of the representation $(\pi_0 \rtimes (V_0 \otimes \overline{U_0})) \circ \iota = (\pi^{\otimes n} \rtimes (W(\pi) \otimes \overline{\lambda})) \circ \iota$. This nondegenerate part is unitarily equivalent to the representation

$$S^n(\mathcal{A}) = (\mathcal{A}^{\otimes n})^{S_n} \rightarrow B(\text{Hom}_{S_n}(\mathcal{V}, \mathcal{H}^{\otimes n})), \quad x \mapsto \pi^{\otimes n}(x) \otimes \text{id}_{\mathcal{V}^*} \quad (11)$$

by Lemma 5.5(ii), since

$$(\pi^{\otimes n}(x) \otimes \text{id}_{\mathcal{V}^*})(T) = \pi^{\otimes n}(x)T \text{ for every } T \in \text{Hom}_{S_n}(\mathcal{V}, \mathcal{H}^{\otimes n}) \simeq (\mathcal{H}^{\otimes n} \otimes \mathcal{V}^*)^{S_n}.$$

But then, by composing (11) with the power map $U(\mathcal{A}) \rightarrow S^n(\mathcal{A}), a \mapsto a^{\otimes n}$ one obtains a unitary irreducible representation of $U(\mathcal{A})$ (compare [BN15, Cor. 6.10]) on the Hilbert space $\text{Hom}_{S_n}(\mathcal{V}, \mathcal{H}^{\otimes n}) =: \mathbb{S}_\lambda(\mathcal{H}^{\otimes n})$ which is unitarily equivalent to the Schur–Weyl representation $\pi_\lambda^{\mathcal{A}}$ (see [BN12, Rem. A.6]). \square

In Assertion (ii) of the following statement we use the notation $\mathcal{C}_1(\mathcal{A}^{\otimes n}, S_n, \alpha)$ as introduced in Example 4.17.

Theorem 5.8. *For any C^* -algebra \mathcal{A} , the following assertions hold:*

- (i) *The Schur–Weyl map $\Psi_{\mathcal{A}}$ from (10) is injective.*
- (ii) *One has $\text{Im } \Psi_{\mathcal{A}} \subseteq \bigcup_{n \geq 0} S^n(\mathcal{A})^\wedge \hookrightarrow \text{U}(\mathcal{A})^\wedge$ and $\text{Im } \Psi_{\mathcal{A}} \cap S^n(\mathcal{A})^\wedge$ consists of restrictions of representations which belong to $\mathcal{C}_1(\mathcal{A}^{\otimes n}, S_n, \alpha)$, for all $n \geq 1$.*
- (iii) *The map $\Psi_{\mathcal{A}}$ is bijective if \mathcal{A} is isomorphic to the C^* -algebra of compact operators on some complex Hilbert space.*
- (iv) *If $\pi_j: \mathcal{A} \rightarrow B(\mathcal{H}_j)$, $j = 1, 2$, are inequivalent irreducible $*$ -representations, then the map $a \mapsto \pi_1(a) \otimes \pi_2(a)$ is a unitary irreducible representation of $\text{U}(\mathcal{A})$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ whose equivalence class does not belong to the image of the Schur–Weyl map.*

Proof. (i), (ii) Recall from (10) that the Schur–Weyl map is

$$\Psi_{\mathcal{A}}: \bigsqcup_{[\pi] \in \widehat{\mathcal{A}}} \mathcal{P}_{J_{[\pi]}}/S_{(J_{[\pi]})} \rightarrow \text{U}(\mathcal{A})^\wedge, \quad [\lambda] \mapsto [\pi_\lambda^{\mathcal{A}}] \quad \text{for } [\lambda] \in \mathcal{P}_{J_{[\pi]}}/S_{(J_{[\pi]})} \text{ and } [\pi] \in \widehat{\mathcal{A}}.$$

To prove that $\Psi_{\mathcal{A}}$ is injective, let $[\pi_j] \in \widehat{\mathcal{A}}$ and $\lambda_j \in \mathcal{P}_{J_{[\pi_j]}}$ for $j = 1, 2$ with $[(\pi_1)_{\lambda_1}^{\mathcal{A}}] = [(\pi_2)_{\lambda_2}^{\mathcal{A}}] \in \text{U}(\mathcal{A})^\wedge$. We must check that $[\pi_1] = [\pi_2] := [\pi] \in \widehat{\mathcal{A}}$ and $[\lambda_1] = [\lambda_2] \in \mathcal{P}_{J_{[\pi]}}/S_{(J_{[\pi]})}$.

Denote also by $\lambda_j: S_n \rightarrow B(\mathcal{V}_j)$ the unitary irreducible representation associated with $\lambda_j \in \mathcal{P}_{J_{[\pi]}}$, where $n \geq \max\{|\text{supp } \lambda_1|, |\text{supp } \lambda_2|\}$. It follows by Proposition 5.7 that $[\pi_\lambda] \in S^n(\mathcal{A})^\wedge$ is the image through the bijection \mathcal{I} (Theorem 4.23) of the irreducible covariant representation $((\pi_j^{\otimes n})^\nu, W(\pi_j) \otimes \overline{\lambda_j})$ of $(\mathcal{A}^{\otimes n}, S_n, \alpha)$. Then $[(\pi_1)_{\lambda_1}^{\mathcal{A}}] = [(\pi_2)_{\lambda_2}^{\mathcal{A}}] \in S^n(\mathcal{A})^\wedge \hookrightarrow \widehat{\mathcal{A}}$, implies

$$[(\pi_1^{\otimes n})^{\nu_1} \rtimes (W(\pi_1) \otimes \overline{\lambda_1})] = [(\pi_2^{\otimes n})^{\nu_2} \rtimes (W(\pi_2) \otimes \overline{\lambda_2})] \in (\mathcal{A}^{\otimes n} \rtimes S_n)^\wedge.$$

Now we can use Example 4.17 and the injectivity of the map (9) in Lemma 5.2 to obtain $[\pi_1] = [\pi_2]$, that is, $[\pi_1] = [\pi_2] := [\pi] \in \widehat{\mathcal{A}}$. It then follows by [BN12, Th. 4.4] that also $[\lambda_1] = [\lambda_2] \in \mathcal{P}_{J_{[\pi]}}/S_{(J_{[\pi]})}$, and this completes the proof of the fact that $\Psi_{\mathcal{A}}$ is injective.

(iii) If \mathcal{A} is isomorphic to the C^* -algebra of compact operators on some complex Hilbert space, then $\Psi_{\mathcal{A}}$ is surjective by [Ne14, Th. 3.21].

(iv) Now let $\pi_j: \mathcal{A} \rightarrow B(\mathcal{H}_j)$ for $j = 1, 2$ be inequivalent irreducible $*$ -representations and define the irreducible $*$ -representation $\pi_0 := \pi_1 \otimes \pi_2: \mathcal{A}^{\otimes 2} \rightarrow B(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Using again the injectivity of the map (9) in Lemma 5.2 it follows that the isotropy group of $[\pi_0]$ for the natural action of S_2 on $\widehat{\mathcal{A}^{\otimes 2}}$ is $\{1\}$, that is, $[\pi]$ belongs to the class $\mathcal{C}_0(\mathcal{A}^{\otimes 2}, S_2, \alpha)$ from Example 4.16. This shows that the image of $[\pi_0]$ under \mathcal{I} does not belong to the image of the Schur–Weyl map $\Psi_{\mathcal{A}}$. It only remains to compute the image of $[\pi_0]$ under the map \mathcal{I} . To this end we first compute the corresponding induced representation (π, U) of $(\mathcal{A}^{\otimes 2}, S_2, \alpha)$ given by Definition 4.1. Since $G_0 = \{1\}$, we have $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0$ thought of as column vectors and

$$U: S_2 \simeq \mathbb{Z}_2 \rightarrow B(\mathcal{H}) \text{ with } U_0 = \mathbf{1} \text{ and } U_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and also

$$\pi: \mathcal{A}^{\otimes 2} \rightarrow B(\mathcal{H}), \quad \pi(a_1 \otimes a_2) = \begin{pmatrix} \pi_1(a_1) \otimes \pi_2(a_2) & 0 \\ 0 & \pi_1(a_2) \otimes \pi_2(a_1) \end{pmatrix}$$

for all $a_1, a_2 \in \mathcal{A}$. If we view the elements of the crossed product $\mathcal{A}^{\otimes 2} \rtimes S_2$ as pairs of elements of $\mathcal{A}^{\otimes 2}$ with the multiplication given by

$$\begin{aligned} (a_1 \otimes a_2, b_1 \otimes b_2) \cdot (c_1 \otimes c_2, d_1 \otimes d_2) \\ = ((a_1 c_1) \otimes (a_2 c_2) + (b_1 d_1) \otimes (b_2 d_2), (a_1 d_1) \otimes (a_2 d_2) + (b_1 c_2) \otimes (b_2 c_1)) \end{aligned}$$

then the irreducible representation $\pi \rtimes U: \mathcal{A}^{\otimes 2} \rtimes S_2 \rightarrow B(\mathcal{H})$ is given by

$$\begin{aligned} (\pi \rtimes U)(a_1 \otimes a_2, b_1 \otimes b_2) &= \pi(a_1 \otimes a_2)U_0 + \pi(b_1 \otimes b_2)U_1 \\ &= \begin{pmatrix} \pi_1(a_1) \otimes \pi_2(a_2) & \pi_1(b_2) \otimes \pi_2(b_1) \\ \pi_1(b_1) \otimes \pi_2(b_2) & \pi_1(a_2) \otimes \pi_2(a_1) \end{pmatrix} \end{aligned}$$

and by composing $\pi \rtimes U$ with $\iota: S^2(\mathcal{A}) \rightarrow \mathcal{A}^{\otimes 2} \rtimes S_2$, $\iota(a \otimes a) = (a \otimes a, a \otimes a)$, we obtain a degenerate representation on \mathcal{H} whose essential subspace is the image of $\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}$, i.e., the “diagonal subspace” of $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0$. The nondegenerate part of $(\pi \rtimes U) \circ \iota$ is unitarily equivalent to the representation

$$S^2(\mathcal{A}) \rightarrow B(\mathcal{H}_0), \quad a \otimes a \mapsto \pi_1(a) \otimes \pi_2(a)$$

(and is irreducible by Proposition 4.21(iii)). Composing it further with the embedding $U(\mathcal{A}) \rightarrow S^2(\mathcal{A})$, $a \mapsto a \otimes a$, we obtain the irreducible representation of $U(\mathcal{A})$ from the statement. This concludes the proof. \square

6 Schur–Weyl theory for factor representations

In Propositions 6.1–6.2 below we give a generalization of [Nes13, Th. 1(1)] dealing with factors of type II_1 to more general von Neumann algebras, and this also provides a partial generalization of [EnIz15, Th. 4.1] where a Schur–Weyl duality property was established for the standard representation of the hyperfinite factor of type II_1 . In the case $\mathcal{M} = B(\mathcal{H})$ the following proposition also recovers the Schur–Weyl duality studied in [BN12].

Proposition 6.1. *Let $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra and for any $n \geq 1$ consider the unitary permutation representation $V: S_n \rightarrow B(\mathcal{H}^{\otimes n})$ and its corresponding action by $*$ -automorphisms $\alpha: S_n \rightarrow \text{Aut}(\mathcal{M}^{\otimes n})$. Then for the homomorphism of multiplicative $*$ -semigroups $\Gamma: \mathcal{M} \rightarrow (\mathcal{M}^{\otimes n})^{S_n}$, $\Gamma(a) := a^{\otimes n}$, one has*

$$\Gamma(U(\mathcal{M}))'' = (\mathcal{M}^{\otimes n})^{S_n} \quad \text{and} \quad \Gamma(U(\mathcal{M}))' = (V_{S_n} \cup (\mathcal{M}')^{\otimes n})''.$$

Proof. The inclusion $\Gamma(U(\mathcal{M}))'' \subseteq (\mathcal{M}^{\otimes n})^{S_n}$ is clear. For the opposite inclusion we use the differential of Γ at $0 \in \mathcal{M}$,

$$d\Gamma: \mathcal{M} \rightarrow (\mathcal{M}^{\otimes n})^{S_n}, \quad d\Gamma(a) = \sum_{k=1}^n \mathbf{1}^{\otimes(k-1)} \otimes a \otimes \mathbf{1}^{\otimes(n-k)}.$$

By the formulas $\mathrm{d}\Gamma(a) = \frac{1}{i} \frac{d}{dt} \big|_{t=0} \Gamma(e^{ita})$ and $\Gamma(e^{ia}) = e^{i\mathrm{d}\Gamma(a)}$ for all $a = a^* \in \mathcal{M}$, one easily obtains the equality $\Gamma(\mathrm{U}(\mathcal{M}))'' = \mathrm{d}\Gamma(\mathcal{M})''$, hence it suffices to prove that $(\mathcal{M}^{\otimes n})^{S_n} \subseteq \mathrm{d}\Gamma(\mathcal{M})''$. As one has the surjective map

$$E: \mathcal{M}^{\otimes n} \rightarrow (\mathcal{M}^{\otimes n})^{S_n}, \quad E(b) = \frac{1}{n!} \sum_{\sigma \in S_n} \alpha_\sigma(b)$$

we will have to prove

$$(\forall b_1, \dots, b_n \in \mathcal{M}) \quad E(b_1 \otimes \dots \otimes b_n) \in \mathrm{d}\Gamma(\mathcal{M})''. \quad (12)$$

We will prove this by recurrence after $r := |\{j \in \{1, \dots, n\} \mid b_j \neq \mathbf{1}\}|$. If $r = 1$, this is clear since $E(b_1 \otimes \dots \otimes b_n) = \frac{1}{n!} \mathrm{d}\Gamma(b_{j_1})$, where j_1 is the unique $j \in \{1, \dots, n\}$ with $b_j \neq \mathbf{1}$. Now assume $r + 1 \leq n$ and define $\ell_0 := \max\{j \in \{1, \dots, n\} \mid b_j \neq \mathbf{1}\}$, hence $b_{\ell_0} \neq \mathbf{1}$. Also define $c_j := b_j$ if $j \in \{1, \dots, n\} \setminus \{\ell_0\}$ and $c_{\ell_0} := \mathbf{1}$. Then one has

$$\begin{aligned} \mathrm{d}\Gamma(b_{\ell_0}) \cdot n! E(c_1 \otimes \dots \otimes c_n) &= \left(\sum_{k=1}^n \mathbf{1}^{\otimes(k-1)} \otimes b_{\ell_0} \otimes \mathbf{1}^{\otimes(n-k)} \right) \left(\sum_{\sigma \in S_n} c_{\sigma(1)} \otimes \dots \otimes c_{\sigma(n)} \right) \\ &= \sum_{k=1}^n \sum_{\sigma \in S_n} c_{\sigma(1)} \otimes \dots \otimes (b_{\ell_0} c_{\sigma(k)}) \otimes \dots \otimes c_{\sigma(n)} \end{aligned}$$

If we split the above sum according to the pairs $(\sigma, k) \in S_n \times \{1, \dots, n\}$ which satisfy $\sigma(k) = \ell_0$ and $\sigma(k) \neq \ell_0$, respectively, then we obtain

$$\begin{aligned} \mathrm{d}\Gamma(b_{\ell_0}) \cdot n! E(c_1 \otimes \dots \otimes c_n) &= n! E(b_1 \otimes \dots \otimes b_n) \\ &\quad + n! \sum_{k \in \{1, \dots, n-1\} \setminus \{\ell_0\}} E(b_1 \otimes \dots \otimes (b_{\ell_0} b_k) \otimes \dots \otimes b_n) \end{aligned}$$

where in every summand of the second sum, the ℓ_0 -th factor in the tensor product is equal to $\mathbf{1}$. Solving the above equation for $E(b_1 \otimes \dots \otimes b_n)$ and using the recurrence hypothesis, one completes the proof of (12), and we have already seen above that this implies the assertion.

It remains to prove the second equality from the statement. Using the commutation theorem for von Neumann algebras $(\mathcal{M}_1 \otimes \mathcal{M}_2)' = \mathcal{M}_1' \otimes \mathcal{M}_2'$ and the Bicommutant Theorem, one obtains

$$(V_{S_n} \cup (\mathcal{M}')^{\otimes n})' = (V_{S_n} \cup (\mathcal{M}^{\otimes n})')' = V_{S_n}' \cap \mathcal{M}^{\otimes n} = (\mathcal{M}^{\otimes n})^{S_n} = \Gamma(\mathrm{U}(\mathcal{M}))''$$

where we used also the first of the asserted equalities from the statement, which was already proved. Now the second equality from the statement follows by taking commutants in the above equalities and using once again the Bicommutant Theorem. \square

The following proposition shows that, beyond the classical Schur–Weyl theory when $\mathcal{M} = B(\mathcal{H})$, the picture is completely different.

Proposition 6.2. *Assume the setting of Proposition 6.1. If \mathcal{M} is a factor of type II_1 , II_∞ , or III , then $\Gamma: \mathrm{U}(\mathcal{M}) \rightarrow B(\mathcal{H}^{\otimes n})$, $\Gamma(u) = u^{\otimes n}$, is a factor representation of the same type as \mathcal{M} .*

Proof. By Proposition 6.1 it suffices to prove that $(\mathcal{M}^{\otimes n})^{S_n}$ is a factor of the same type as \mathcal{M} .

Since \mathcal{M} is not a factor of type I, it follows by [Sa75, Th. 5] that for every $\sigma \in S_n \setminus \{1\}$ the automorphism $\alpha(\sigma)$ of $\mathcal{M}^{\otimes n}$ is not inner, that is, it is not of the form $x \mapsto u x u^*$ for any unitary element $u \in \mathcal{M}$. Then, since \mathcal{M} is a factor, it follows by [Sa71, Prop. 2.6.7] that $\mathcal{M}^{\otimes n}$ is a factor, hence it follows by [Au76, Cor. to Prop. II.3] that $(\mathcal{M}^{\otimes n})^{S_n}$ is in turn a factor.

To check that $(\mathcal{M}^{\otimes n})^{S_n}$ has the same type as \mathcal{M} , we discuss below separately the cases that can occur, using the faithful normal conditional expectation

$$E: \mathcal{M}^{\otimes n} \rightarrow (\mathcal{M}^{\otimes n})^{S_n}, \quad E(a) = \frac{1}{n!} \sum_{\sigma \in S_n} \alpha_\sigma(a).$$

If $0 \leq a \in \mathcal{M}^{\otimes n}$ then, using the fact that $0 \leq \alpha_\sigma(a)$ if $\sigma \in S_n \setminus \{1\}$ and $a = \alpha_\sigma(a)$ if $\sigma = 1$, one obtains $a \leq n!E(a)$, hence the conditional expectation E has finite index in the sense of [FrKi98]. It then follows by [FrKi98, Prop. 2.2] that the factors $\mathcal{M}^{\otimes n}$ and $(\mathcal{M}^{\otimes n})^{S_n}$ have the same type II_1 , II_∞ , or III .

If \mathcal{M} is of type III , then $\mathcal{M}^{\otimes n}$ is of type III by [Sa71, Th. 2.6.4]. If \mathcal{M} is of type II_1 or II_∞ , then $\mathcal{M}^{\otimes n}$ is of the same type by [Sa71, Prop. 2.6.1/3]. This completes the proof. \square

Remark 6.3. With the notation of the proof of Proposition 6.2, the index of the conditional expectation E is $\text{Ind } E = |S_n| = n!$ by [Lo92, Ex. 1.2]. Moreover, if \mathcal{M} is of type III_λ for some $\lambda \in [0, 1]$, then more refined information on the type of $(\mathcal{M}^{\otimes n})^{S_n}$ can be obtained by [Lo92, Th. 2.7].

The following theorem extends [Nes13, Th. 1(2)] to infinite factors.

Theorem 6.4. *In the setting of Proposition 6.2 we define for every non-zero projection $p = p^2 = p^* \in \Gamma(\text{U}(\mathcal{M}))'$ with range $\mathcal{H}_p = p\mathcal{H}$ the representation*

$$\Gamma_p: \text{U}(\mathcal{M}) \rightarrow \text{U}(\mathcal{H}_p), \quad \Gamma_p(u) := \Gamma(u)|_{\mathcal{H}_p}.$$

- (i) Γ_p is a factor representation of the same type II or III as \mathcal{M} and $\Gamma_p \approx \Gamma$.
- (ii) If \mathcal{M} is a factor of type II, then for any faithful normal trace τ on $\Gamma(\text{U}(\mathcal{M}))'$ and finite projections $p_j = p_j^2 = p_j^* \in \Gamma(\text{U}(\mathcal{M}))'$ for $j = 1, 2$, one has $\Gamma_{p_1} \leq \Gamma_{p_2}$ if and only if $\tau(p_1) \leq \tau(p_2)$, and $\Gamma_{p_1} \simeq \Gamma_{p_2}$ if and only if $\tau(p_1) = \tau(p_2)$.
- (iii) If \mathcal{M} is a countably decomposable factor of type III, then $\Gamma_p \simeq \Gamma$.

Proof. (i) It is easily seen that $\Gamma_p(\text{U}(\mathcal{M}))'' = \Gamma(\text{U}(\mathcal{M}))_p$ (reduced von Neumann algebra), hence Γ_p is a factor representation, by [SZ79, Th. 3.13]. Moreover, the reduction map $\Gamma(\text{U}(\mathcal{M})) \rightarrow \Gamma(\text{U}(\mathcal{M}))_p$ is a $*$ -isomorphism by [SZ79, Prop. 3.14], hence Γ and Γ_p are quasi-equivalent factor representations.

(ii) If \mathcal{M} is a factor of type II, then $\Gamma(\text{U}(\mathcal{M}))''$ is a factor of type II by Proposition 6.2, hence by [Dix69, Ch. I, §6, no. 8, Cor. 1 of Prop. 13] also $\Gamma(\text{U}(\mathcal{M}))'$ is a factor of type II, and then it has a faithful normal trace τ . If $p_j = p_j^2 = p_j^* \in \Gamma(\text{U}(\mathcal{M}))'$ are finite projections for $j = 1, 2$, then one has $\tau(p_1) \leq \tau(p_2)$ (respectively $\tau(p_1) = \tau(p_2)$) if and only if $p_1 \prec p_2$

(respectively $p_1 \sim p_2$) in $\Gamma(U(\mathcal{M}))'$ by [Dix69, Ch. III, §2, Prop. 13], which is further equivalent to $\Gamma_{p_1} \leq \Gamma_{p_2}$ by [Dix64, Cor. 5.1.4] (respectively $\Gamma_{p_1} = \Gamma_{p_2}$ by [Dix64, Cor. 5.1.3]).

(iii) If \mathcal{M} is a countably decomposable factor of type III and $0 \neq p_j = p_j^2 = p_j^* \in \Gamma(U(\mathcal{M}))'$ for $j = 1, 2$, then $p_1 \sim p_2$ in $\Gamma(U(\mathcal{M}))'$ by [Sa71, Prop. 2.2.14], hence Γ_{p_1} and Γ_{p_2} are unitarily equivalent again by [Dix64, Cor. 5.1.3]. \square

Remark 6.5. It follows by Proposition 6.1 that to every central projection $\chi \in \mathbb{C}[S_n]$ there corresponds an orthogonal projection $V(\chi) \in \Gamma(U(\mathcal{M}))'$. In the case when \mathcal{M} is a factor of type II, the traces of these projections were computed in the proof of [Nes13, Th. 1(3)] and in [DaKa06, Cor. 5].

When \mathcal{M} is a factor of type II_1 with its faithful normal tracial state τ , the character of the representation Γ is $(\tau|_{U(\mathcal{M})})^n$ (which belongs to the list in [EnIz15, Th. 1.5]) and this is equal to the character of the representation Γ_p if $0 \neq p = p^2 = p^* \in \Gamma(U(\mathcal{M}))'$, because, by Theorem 6.4, the representations Γ and Γ_p are quasi-equivalent.

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